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On Fibred Spaces with Invariant Equiform Connection and Torseforming Structure Vector Field

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Geometria differenziale. — *On Fibred Spaces with Invariant Equiform Connection and Torseforming Structure Vector Field.* Nota di TANJIRO OKUBO, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono discusse condizioni necessarie e sufficienti affinché uno spazio fibrato con fibra 1-dimensionale sia tale che un vettore tangente alla fibra risulti torsiforme, in modo inoltre che lo spazio ammetta una connessione invariante equiforme. Si mostra poi come uno spazio fibrato tale che ad ogni cammino sullo spazio base corrisponda per proiezione ancora un cammino, possa venire dotato di una connessione invariante equiforme coll'assegnare un'arbitraria connessione affine simmetrica sullo spazio base.

INTRODUCTION

The notion of fibred spaces has recently been developed by many geometers (1). In differential-geometrical point of view its idea goes far back to the five dimensional metric space considered by Th. Kaluza [2] and O. Klein [3] for establishing a unified field theory of gravitation and electromagetism. In those fibred spaces it has been known that there is a notable fibre structure such that the projection of any path in total space is still a path in base space with respect to the connection induced from an invariant connection introduced in total space [1]. On the other hand a special connection called the volume preserving connection or simply the equiform connection that arises in the theory of affine connections serves best for dealing with affine metric properties parallel to the rôle of the Levi-Civita connection in Riemannian manifolds. In this paper we shall deal with these two concepts by taking fibred spaces with 1-dimensional fibre in which the vector field tangent to each fibre is torseforming and the total space admits an invariant equiform connection.

§ 1 is devoted to describe the structure of fibred spaces with 1-dimensional fibre and one sees there that the torseforming vector field indeed provides the fibred space with the fibre structure said as notable above (Proposition 2). §* 2 discusses the condition for such a fibred space to admit an invariant equiform connection and shows for this kind of fibred spaces that if the horizontal lift of a path in the base space is also a path in the total space, then it is always possible to endow an invariant equiform connection to the total space by giving an arbitrary symmetric affine connection in the base space.

(*) Nella seduta dell'8 febbraio 1975.

(1) For the development of the theory one refers to those papers almost comprehensively enlisted in the bibliographical section of " Differential geometry of fibred space " (S. Ishihara and M. Konishi [1]). Throughout the present paper we adopt the same terminologies and notations that were used in [1].

§ 1. Fibred affine spaces and torse-forming vector field C

Let $\tilde{M}$ and M be two differentiable manifolds, of dimension $n+1$ and n respectively and assume that there is a differentiable mapping $\pi : \tilde{M} \to M$ which is onto and of maximum rank n everywhere. Then, for each point P of M, its inverse image $\pi^{-1}(P)$ is a curve in $\tilde{M}$ which is called the *fibre* over P. We suppose that every fibre is not discrete. Such a set $\{\tilde{M}, M, \pi\}$ is called a *fibred space*, M the *total space*, $\tilde{M}$ the *base space* and π the *projection*. Throughout the present paper, manifolds, mapping and geometric objects we deal with are assumed to be differentiable and of class C^{∞}. Let C be a vector field tangent to fibres such that C is non-zero everywhere in $\tilde{M}$. Let η be a 1-form satisfying

$$\eta(C) = 1 \quad , \quad \mathcal{L}\eta = 0 , \tag{1.1}$$

where $\mathcal{L}$ denotes the Lie derivative with respect to C. For a vector field $\tilde{X}$ in $\tilde{M}$, we define its horizontal part $\tilde{X}^H$ by

$$\tilde{X}^H = \tilde{X} - \eta(\tilde{X})\, C ,$$

and for *a* 1-form $\tilde{\omega}$ in $\tilde{M}$, its horizontal part $\tilde{\omega}^H$

$$\tilde{\omega}^H = \tilde{\omega} - \tilde{\omega}(C)\, \eta .$$

$\eta(X)\,C$ and $\tilde{\omega}(C)\,\eta$ are then called the vertical part of X and $\tilde{\omega}$ respectively. For a function $\tilde{f}$ in $\tilde{M}$, its horizontal part $\tilde{f}$ is identified with f itself. For a tensor field $\tilde{T}$, its horizontal part $\tilde{T}^H$ and vertical part $\tilde{T}^V$ are tensor fields of the same type as $\tilde{T}$, such that they are inductively characterized by the formula

$$(\tilde{S} \otimes \tilde{T})^H = \tilde{S} \otimes \tilde{T}^H \quad , \quad (\tilde{S} + \tilde{U})^H = \tilde{S}H + \tilde{U}^H ,$$
$$(\tilde{S} \otimes \tilde{T})^V = \tilde{S} \otimes \tilde{T}^V \quad , \quad (\tilde{S} + \tilde{U})^V = \tilde{S}V + \tilde{U}^V .$$

$\tilde{T}$ is said to be invariant if it satisfies the condition

$$\mathcal{L}\tilde{T} = 0$$

and to be projectable if it satisfies

$$(\mathcal{L}\tilde{T}^H)^H = 0 .$$

Let $\{\tilde{U}, X^h\}$ (2) and $\{\tilde{U}, v^a\}$ be local coordinate neighbourhoods of $\tilde{M}$ and M respectively such that $U = \pi(\tilde{U})$. The projection $\pi : \tilde{M} \to M$ is locally expressed by certain equations

$$v^a = v^a(x^h) \tag{1.2}$$

(2) We adopt the following convention for inidices: h, i, j, k run over the range $1, 2, \cdots, n+1$, and a, b, c, d, e run over the range $1, 2, \cdots, n$.

in $\{\tilde{U}, x^h\}$ and $\{U, v^a\}$, where x^h are coordinates of an arbitrary point $\tilde{P} \in \tilde{U}$ and v^a are those of the point $P = \pi \tilde{P} \in U$, $v^a(x^h)$ being differentiable functions of variables x^h. Differentiating (1.2) we put

$$E_i^a = \partial_i v^a \quad , \quad (\partial_i = \partial/\partial x^i). \tag{1.3}$$

Then

$$\partial_j E_i^a = \partial_i E_j^a, \tag{1.4}$$

and we have the local covector fields E^a having in $\tilde{U}$ with components E_i^a for each index a. Since C is tangent to each fibre and $\eta(C) = 1$,

$$C^i C_i = 1 \quad , \quad C_i E_i^a = 0 \tag{1.5}$$

where C_i are the components of η. These $n+1$ local covector fields E^a and η are linearly independent and hence form a coframe in $\tilde{U}$. Because of (1.5) the inverse of the matrix (E_i^a, C_i) has the form

$$(E_i^a, C_i)^{-1} = \begin{bmatrix} E^h{}_b \\ C^h \end{bmatrix}$$

which gives rise n local vector field E_b in $\tilde{U}$ having the component $E^h{}_b$, and we have

$$E^i{}_b D_i^a = \delta_b^a \quad , \quad E^i{}_b C_i = 0 \quad , \quad C^i E_i^b = 0 \quad , \quad C^i C_i = 1\,, \tag{1.6}$$

that is,

$$\mathscr{L}E^a(E_b) = \delta_b^a \quad , \quad \eta(E_b) = 0 \quad , \quad E^a(C) = 0 \quad , \quad \eta(C) = 1\,,$$

or equivalently

$$E_i^a E^h{}_a + C_i C^h = \delta_i^h\,. \tag{1.7}$$

Because of (1.4) and (1.5) we have

$$\mathscr{L}E_i^a = C^j \partial_j E_i^a + E_i^a \partial_i C^j = C^j \partial_i E_j^a + E_j^a \partial_i C^j = C_i (C^j E_j^a) = 0\,,$$

which together with (1.1) and $\mathscr{L}C = 0$ yields the formula

$$\mathscr{L}E_a = 0 \quad , \quad \mathscr{L}C = 0 \quad , \quad \mathscr{L}E^a = 0 \quad , \quad \mathscr{L}\eta = 0\,. \tag{1.8}$$

A function $\tilde{f}$ in $\tilde{M}$ is invariant if and only if $\tilde{f}$ is constant along each fibre and there is a unique function f in M given by $\tilde{f} = f \circ \pi$. Any tensor field in $\tilde{M}$, say $\tilde{T}$ of type (1.1), is expressed locally in $\tilde{U}$ as

$$\tilde{T} = T_b{}^a E^b \otimes E_a + T_b{}^0 E^b \otimes C + T_0{}^a \eta \otimes E_a + T_0{}^0 \eta \otimes C\,. \tag{1.9}$$

$T_b{}^a, T_b{}^0, T_0{}^a$ and $T_0{}^0$ are functions in $\tilde{U}$. Then if $\tilde{T}$ is invariant, then $T_b{}^a$ is an invariant function, i.e. $\mathscr{L}T_b{}^a = 0$. $\tilde{T}$ is projectable if and only if $T_b{}^a$ is an invariant function and for any such projectable tensor field $\tilde{T}$ of $\tilde{M}$ there is a unique tensor field T in M, called the projection of $\tilde{T}$ and denoted by $T = p\tilde{T}$, such that T has components $T_b{}^a$ in $\tilde{U}$ if $\tilde{T}$ has the local expression (1.9) in $\tilde{U}$. Given a tensor field in M, say T of type (1.1) with component $T_b{}^a$ in $\{U, v^a\}$, a horizontal tensor field $\tilde{T}$ in $\tilde{M}$ will be defined by the local expression

$$\tilde{T} = T_b{}^a E^b \otimes E_a$$

in $\tilde{U}$ and $\tilde{T}$ is called the lift of T and denoted by $\tilde{T} = T^L$. For a function f in M, its lift f^L is defined by $f^L = f \circ \pi$.

An affine connection $\tilde{\nabla}$ is called an invariant affine connection if $\mathscr{L}\tilde{\nabla} = 0$, that is, if $\tilde{\nabla}$ satisfies

$$\mathscr{L}(\tilde{\nabla}_{\tilde{Y}} \tilde{X}) = \tilde{\nabla}_{\tilde{Y}} \mathscr{L}\tilde{X} + \nabla_{[C,\tilde{Y}]} \tilde{X}, \tag{1.10}$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ in $\tilde{M}$. Taking X^L and Y^L of arbitrary vector fields X and Y in M and substituting them into (1.10) we find

$$\mathscr{L}(\tilde{\nabla}_{Y^L} X^L) = 0$$

which shows that $\tilde{\nabla}_{Y^L} X^L$ is an invariant vector and is projectable. Hence

$$p(\tilde{\nabla}_{Y^L} X^L) = \nabla_Y X \tag{1.11}$$

defines an affine connection ∇ in M, which we call the projection of $\tilde{\nabla}$ and denote by $p\tilde{\nabla}$. Supposing that such invariant connection is torsionless, we assume that the vector field C satisfies the condition

$$\tilde{\nabla}_{\tilde{Y}} C = \alpha \tilde{Y} \quad , \quad (\alpha \neq 0 \text{ is constant}). \tag{1.12}$$

In this case C is said to be a torseforming vector field. Then, by means of (1.6) and $\mathscr{L}\eta = 0$, we have (1)

$$\begin{aligned}
&\tilde{\nabla}_j E_i{}^a = -\Gamma^a_{cb} E_j{}^c E_i{}^b - \alpha E_j{}^a C_i - \alpha C_j E_i{}^a, \\
&\tilde{\nabla}_j C_i = -h_{cb} E_j{}^c E_i{}^b - \alpha C_j C_i, \\
&\tilde{\nabla}_j E^h{}_b = \Gamma^a_{cb} E_j{}^c E^h{}_a + h_{cb} E_j{}^c C^h + \alpha C_j E^h{}_b, \\
&\tilde{\nabla}_j C^h = \alpha \delta_j^h,
\end{aligned} \tag{1.13}$$

where Γ^a_{cb} and h_{cb} are invariant functions locally defined in $\tilde{U}$. Since (1.4) is identical with the equation

$$\tilde{\nabla}_j E_i{}^a = \tilde{\nabla}_i E_j{}^a,$$

we see from the first of the equations in (1.13) that

$$\Gamma^a_{cb} = \Gamma^a_{bc}, \tag{1.14}$$

that is, the connection ∇ of M defined by (1.11) is torsionless.

Tensor field $\tilde{h}$ with local espression

$$\tilde{h} = h_{cb}\, E^c \otimes E^b$$

in $\tilde{U}$ determines a global tensor field in $\tilde{U}$ and its projection $p\tilde{h}$ is now denoted by h. Because of the second equation in (1.13) we have

$$d\eta = -\frac{1}{2}(h_{cb} - h_{bc})\, E_j^{\;c} E_i^{\;b}\, dx^j \wedge dx^i \tag{1.15}$$

and, since d d$\eta = 0$, we have

$$\partial_d (h_{cb} - h_{bc}) + \partial_c (h_{bd} - h_{db}) + \partial_b (h_{dc} - h_{cd}) = 0 \tag{1.16}$$

or by using (1.14)

$$\nabla_d (h_{cb} - h_{bc}) + \nabla_c (h_{bd} - h_{db}) + \nabla_b (h_{dc} - h_{dc}) = 0\,, \tag{1.17}$$

which show that the 2-form $h = h_{cb}\, dv^c \wedge dv^b$ is closed in M and hence h determines the characteristic class of $\{\tilde{M}, M, \pi\}$ (3). In this sense h shall be called the structure tensor field henceforward.

By means of (1.15) the horizontal distribution defined by $\eta = 0$ is integrable if and only if

$$h_{cb} = h_{bc} \tag{1.18}$$

and it is easily seen that $\{\tilde{M}, M, \pi\}$ is locally trivial (1).

In $\tilde{M}$ with connection $\tilde{\nabla}$, a curve Γ in $\tilde{M}$ is called a path if

$$\Gamma'' = a\Gamma\,,$$

a being a function along c. Then the fourth equation in (1.13) implies

$$C^j\, \tilde{\nabla}_j\, C^h = \alpha C^h.$$

Hence we have

PROPOSITION 1. *In a fibred space with invariant affine connection $\tilde{\nabla}$ in which* C *is torseforming, each fibre is a path.*

Let Γ be a curve in $\tilde{M}$ and denote by γ its projection in M. If Γ has a local expression $x^h = x^h(t)$ in $\{\tilde{U}, x^h\}$, t being a parameter, and if γ has a local expression $v^a = v^a(t)$ in $\{U, v^a\}$, then, using (1.2), we have

$$v^a(t) = v^a(x^h(t))$$

from which by differentiation

$$\frac{dv^a}{dt} = E_i^{\;a} \frac{dx^i}{dt}\,.$$

(3) For the existence of the characteristic class determined by h with respect to general fibred spaces, see [1].

Differentiating covariantly and using the first equation in (1.13), we get

$$\frac{\delta^2 v^a}{dt^2} = E_i^{\ a} \frac{\delta^2 x^i}{dt^2} - \alpha C_i \frac{dx^i}{dt} \frac{dv^a}{dt}$$

where

$$\frac{\delta^2 v^a}{dt^2} = \frac{d^2 v^a}{dt^2} + \Gamma^a_{cb} \frac{dv^c}{dt} \frac{dv^b}{dt},$$

$$\frac{\delta^2 x^h}{dt^2} = \frac{d^2 x^h}{dt} + \Gamma^h_{ji} \frac{dx^j}{dt} \frac{dx^i}{dt}$$

and Γ^h_{ji} are the coefficients in $\tilde{U}$ of $\tilde{\nabla}$. Thus we have

PROPOSITION 2. *In the fibred space of Proposition 1 the projection of any path in* $\tilde{M}$ *is a path in* M *with respect to* $\nabla = p\tilde{\nabla}$.

Consider a curve γ in M with local expression $v^a = v^a(t)$, t being a parameter, and denote by Γ the horizontal lift of γ, which is supposed to have local expression $x^h = x^h(t)$. Then

$$\frac{dx^h}{dt} = E^h_{\ a} \frac{dv^a}{dt}.$$

Differentiating covariantly and using the third equation in (1.13), we get

$$\frac{\delta^2 x^h}{dt^2} = E^h_{\ a} \frac{\delta^2 v^a}{dt^2} + \left(h_{cb} \frac{dv^c}{dt} \frac{dv^b}{dt}\right) C^h$$

from which we have

PROPOSITION 3. *If the structure tensor field h satisfies*

$$h_{cb} + h_{bc} = 0, \tag{1.19}$$

then the horizontal lift of a path in M *is also a path in* M *and for the case,* $\pi : \tilde{M} \to M$ *preserves affine parameter along corresponding paths.*

Keeping the assumption that the vector field C of fibred space $\{\tilde{M}, M, \pi\}$ with invariant affine connection $\tilde{\nabla}$ is torseforming and using the Ricci formula

$$\tilde{\nabla}_k \tilde{\nabla}_j \tilde{X}^h - \tilde{\nabla}_j \tilde{\nabla}_k \tilde{X}^h = \tilde{R}_{kji}{}^h \tilde{X}^i \tag{1.20}$$

for any tensor field $\tilde{X} = X^h \partial_h$ in $\tilde{M}$, $\tilde{R}_{kji}{}^h$ being components of curvature tensor $\tilde{R}$ of $\tilde{\nabla}$, we have the structure equations (|)

$$\begin{aligned} \tilde{R}_{dcb}{}^a &= R_{dcb}{}^a + \alpha\,[(\delta^a_d h_{cb} - \delta^a_c h_{db}) - (h_{dc} - h_{cd})\,\delta^a_b] \\ \tilde{R}_{dcb}{}^0 &= \nabla_d h_{cb} - \nabla_c h_{db} \end{aligned} \tag{1.21}$$

$$\begin{aligned} &\tilde{R}_{d0b}{}^0 = 0\,, \quad \tilde{R}_{d0b}{}^0 = 0\,, \quad \tilde{R}_{dc0}{}^a = 0\,, \\ &\tilde{R}_{dc0}{}^0 = 0\,, \quad \tilde{R}_{d00}{}^a = 0\,, \quad \tilde{R}_{d00}{}^0 = 0\,, \end{aligned} \tag{1.22}$$

in virtue of (1.13), where we have put

$$\begin{aligned}
\tilde{R}_{dcb}{}^{a} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, E^{j}{}_{c}\, E^{i}{}_{b}\, E_{h}{}^{a} , & \tilde{R}_{d0b}{}^{a} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, C^{j}\, E^{i}{}_{b}\, E_{h}{}^{a},\\
\tilde{R}_{dcb}{}^{0} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, E^{j}{}_{c}\, E^{i}{}_{b}\, C_{h} , & \tilde{R}_{d0b}{}^{0} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, C^{j}\, E^{i}{}_{b}\, C_{h} ,\\
\tilde{R}_{dc0}{}^{a} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, E^{j}{}_{c}\, C^{i}\, E_{h}{}^{a} , & \tilde{R}_{d00}{}^{a} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, C^{j}\, C^{i}\, E_{h}{}^{a} ,\\
\tilde{R}_{dc0}{}^{0} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, E^{j}{}_{c}\, C^{i}\, C_{h} , & \tilde{R}_{d00}{}^{0} &= \tilde{R}_{kji}{}^{h}\, E^{k}{}_{d}\, C^{j}\, C^{i}\, C_{h}
\end{aligned}$$

and $R_{dcb}{}^{a}$ are components of the curvature tensor R of $\nabla = p\tilde{\nabla}$ defined by the Ricci formula

$$\nabla_d \nabla_c X^a - \nabla_c \nabla_d X^a = R_{dcb}{}^{a} X^b \tag{1.23}$$

for any tensor field $X = X^a \partial_a$, $(\partial_a = \partial/\partial v^a)$, in M.

§ 2. FIBRED SPACE $\{M, M, \pi\}$ WITH INVARIANT EQUIFORM CONNECTION $\tilde{\nabla}$ AND TORSEFORMING VECTOR C.

The affine connection $\tilde{\nabla}$ dealt with in § 1 being torsienless, we have the Bianchi identities of the first and second kind:

$$\tilde{R}_{kji}{}^{h} + \tilde{R}_{jik}{}^{h} + \tilde{R}_{ikj}{}^{h} = 0 , \tag{2.1}$$

$$\tilde{\nabla}_l \tilde{R}_{kji}{}^{h} + \tilde{\nabla}_k \tilde{R}_{jli}{}^{h} + \tilde{\nabla}_l \tilde{R}_{kji}{}^{h} = 0 . \tag{2.2}$$

In (2.1) we put $i = h$ and summing up from 1 to $n + 1$, we have

$$\tilde{R}_{kji}{}^{i} = \tilde{R}_{jk} - \tilde{R}_{kj} , \tag{2.3}$$

where $\tilde{R}_{kj} = \tilde{R}_{ikj}{}^{i}$ are the components in $\tilde{U}$ of the Ricci curvature tensor of $\tilde{\nabla}$ in which we have used the fact resulting from (1.20) that $\tilde{R}_{kji}{}^{h}$ are skew-symmetric with respect to the indices k and j. Again in (2.2) we put $i = h$ and summing up, we get

$$\tilde{\nabla}_l (\tilde{R}_{kj} - \tilde{R}_{jk}) + \tilde{\nabla}_k (\tilde{R}_{jl} - \tilde{R}_{lj}) + \tilde{\nabla}_j (\tilde{R}_{lk} - \tilde{R}_{kl}) = 0 \tag{2.4}$$

in virtue of (2.3).

Similarly we have for $\Delta = p\tilde{\nabla}$,

$$R_{dcb}{}^{a} + R_{cbd}{}^{a} + R_{bdc}{}^{a} = 0$$

$$\nabla_e R_{dcb}{}^{a} + \nabla_d R_{ceb}{}^{a} + \nabla_c R_{ebd}{}^{a} = 0$$

and

$$R_{cba}{}^{a} = R_{bc}{}^{a} - R_{cb}{}^{a} , \tag{2.5}$$

$$\nabla_d (R_{cb} - R_{b0}) + \nabla_c (R_{bd} - R_{db}) + \nabla_b (R_{dc} - R_{cd}) = 0 . \tag{2.6}$$

We now suppose that our invariant affine connection $\tilde{\nabla}$ keeps the affine volume $\mathfrak{u}$ covariantly constant, that is, $\tilde{\nabla}_y \mathfrak{u} = 0$, where $\mathfrak{u}$ is by definition the simple $(n+1)$-vector having components

$$u^{h_1 \cdots h_{n+1}} = \begin{vmatrix} \underset{1}{u}^{h_1} & \cdots & \underset{1}{u}^{h_{n+1}} \\ \cdots & \cdots & \cdots \\ \underset{n+1}{u}^{h_1} & \cdots & \underset{n+1}{u}^{h_{n+1}} \end{vmatrix} \Big/ (n+1)\,!$$

in $\tilde{U}$, and it is characterized by (4)

$$\tilde{R}_{kji}{}^{i} = 0 . \tag{2.7}$$

Such a connection is called the volume preserving affine connection as simply the equiform connection and as is seen from (2.3) the Ricci curvature tensor is symmetric for this case and accordingly the equations (2.4) reduce to trivial ones.

We have assumed that the vector field C is torseforming and obtained the structure equations (1.21) and (1.22). Then the condition (2.7) and (1.22) imply that $\tilde{R}_{dca}{}^{a}$ should vanish, which in turn implies in virtue of (1.21) that we should have

$$R_{dca}{}^{a} = (n+1)\,\alpha\,(h_{dc} - h_{cd}) . \tag{2.8}$$

The comparison of (2.8) with (2.5) yields

$$h_{dc} - h_{cd} = (R_{cd} - R_{dc})/(n+1)\,\alpha . \tag{2.9}$$

Thus we have

THEOREM 1. *Let* $\{\tilde{M}, M, \pi\}$ *be a fibred space with invariant affine connection* $\tilde{\nabla}$ *in which the vector field* C *tangent to the fibre is torseforming. Then the necessary and sufficient condition for* $\tilde{\nabla}$ *to be an equiform connection is that the structure tensor field* h *be related by* (2.9) *with the Ricci curvature tensor field of* $\nabla = p\tilde{\nabla}$. *When this condition is satisfied, equation* (2.4) *coincides with* (1.16).

Further, if we suppose that h is an skew-symmetric tensor, i.e., if (1.19) is satisfied, then we have from (2.9)

$$h_{dc} = (R_{cd} - R_{dc})/\,2\,(n+1)\,\alpha \tag{2.10}$$

and consequently h is completely determined by giving an arbitrary affine connection in M, and the connection $\tilde{\nabla}$ is of equiform because (2.10) yields $\tilde{R}_{kji}{}^{i} = 0$. Hence, by taking account of Proposition 3 too we have

THEOREM 2. *Let* $\{\tilde{M}, M, \pi\}$ *be a fibred space and* ∇ *be an arbitrary symmetric affine connection in* M. *Then, if one chooses the structure tensor* *field h in such a way that h satisfies* (2.10), *it is always possible to endow* $\{\tilde{M}, M, \pi\}$ *with an invariant equiform connection in which the vector field* C *is torseforming and the horizontal lift of any path in* M *is also a path in* $\tilde{M}$ *that has the same affine parameter.*

Coming back to a general $\tilde{\nabla}$, if its Ricci curvature tensor field vanishes identically, then we have (2.7) again in virtue of the equations (2.3) and consequently $\tilde{\nabla}$ is of equiform too. For the case we have from (1.21) and (1.22), [1]

$$h_{dc} = -(nR_{dc} + R_{cd})/(n^2 - 1)\,\alpha\,,$$

which of course satisfies condition (2.9) and in this case h is again completely determined by giving an arbitrary affine connection ∇ in M and for the case it has been proved by S. Ishihara and M. Konishi that the $\tilde{R}^H$ with components $\tilde{R}_{dcb}{}^{a}$ in $\tilde{U}$ given in (1.21) is the Weyl projective curvature, [1].

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