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A series representation for the generalized inverse of a closed linear operator

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Analisi funzionale. — *A series representation for the generalized inverse of a closed linear operator.* Nota di LAWRENCE J. LARDY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si considera un metodo costruttivo per ottenere l'inverso generalizzato di un operatore lineare chiuso densamente definito fra spazi di Hilbert mediante una serie di operatori lineari limitati. Si stabiliscono condizioni per la convergenza della serie e si caratterizzano gli operato i il cui inverso generalizzato risulta compatto.

1. INTRODUCTION

Various aspects of the generalized inverse of an unbounded operator have recently been dealt with from a variety of viewpoints, see [7] for a survey. Here we consider a constructive method for obtaining the generalized inverse of a closed linear operator. If the given operator is bounded, there are a number of methods which can be applied to obtain constructive approximations of its generalized inverse; see, for example [6], [8], [10] and [11]. Also in [6] the method of steepest descent has been applied to obtain the generalized inverse of certain unbounded operators.

We present a representation for the generalized inverse of a densely defined and closed operator between Hilbert spaces as a series of bounded operators. This representation for the generalized inverse of a matrix is noted in [3] and credited to den Broeder and Charnes. For the operators considered here we find that the series converges in the uniform operator topology when the generalized inverse is bounded, that is when the range of the operator is closed, and otherwise it converges pointwise on the domain of the generalized inverse. These convergence results are established in Section 3 where they are then applied to give a characterization of those closed and densely defined linear operators which have compact generalized inverses. This solves a problem posed by Wyler in [16].

2. PRELIMINARIES

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces over the complex field. Consider a closed and densely defined linear operator mapping $\mathscr{H}_1$ to $\mathscr{H}_2$. We denote the null space of A by N (A) and the range of A by R (A). For the adjoint, A^*, of **A** mapping $\mathscr{H}_2$ to $\mathscr{H}_1$, we have $N(A^*) = R(A)^{\perp}$. The orthogonal projection of $\mathscr{H}_1$ onto the closure of $R(A^*)$ is denoted by P and the orthogonal projection of $\mathscr{H}_2$ onto the closure of R (A) is denoted by Q.

(*) Nella seduta dell'8 febbraio 1975.

Under these conditions, there is a generalized inverse $A^\dagger$ of A which is a closed linear operator mapping $\mathscr{H}_2$ to $\mathscr{H}_1$ and satisfies,

$$A^\dagger Ax = Px \quad \text{for} \quad x \in D(A) \tag{2.1}$$

$$AA^\dagger y = Qy \quad \text{for} \quad y \in D(A^\dagger) \tag{2.2}$$

$$D(A^\dagger) = R(A) + R(A)^\perp \tag{2.3}$$

$$N(A^\dagger) = R(A)^\perp \tag{2.4}$$

$$R(A^\dagger) = D(A) \cap N(A)^\perp \tag{2.5}$$

$$AA^\dagger A = A \tag{2.6}$$

$$A^\dagger AA^\dagger = A^\dagger \tag{2.7}$$

(2.8) $A^\dagger$ is continuous if and only if $R(A)$ is closed.

Generalized inverses of densely defined linear operators on Hilbert space were defined and their basic properties were given without proof by Yu. Ya. Tseng in [12], [13], [14] and [15]. We refer the reader to the treatment given by Arghiriade in [1], where conditions (2.1)-(2.5) are developed. The operator $A^\dagger$ coincides with the maximal generalized inverse discussed in [1]. Also, the fact that $A^\dagger$ exists and is a closed linear operator when A is a closed operator is established in [2]. Properties (2.6) and (2.7) follow directly from (2.1)-(2.5). Another approach to the generalized inverse is given by Hestenes in [5] where condition (2.8) is noted.

In [16], Wyler introduces the concept of a Green's operator. In our context, that is for a densely defined closed linear operator mapping the Hilbert space $\mathscr{H}_1$ to $\mathscr{H}_2$, A has a Green's operator if and only if R (A) is closed [16, Theorem 8.4]. It is readily checked that $A^\dagger$ is then a Green's operator for A. Moreover if G is any Green's operator for A, then $D(G) = \mathscr{H}_2$, $AGA = A$, $GAG = G$, AG is a continuous projection of $\mathscr{H}_2$ onto R(A) and GA has a continuous extension to $\mathscr{H}_1$, [16]. Thus if G is a Green's operator for A, then, using (2.6),

$$G = GAG = G(AA^\dagger A)G = (GA)A^\dagger(AG). \tag{2.9}$$

Wyler poses the problem of finding conditions for a Green's operator to be compact. It follows from (2.9) that if $A^\dagger$ is compact, then all the Green's operators for A are compact. In section 3 we give conditions for $A^\dagger$ to be compact and thus solve the problem of Wyler for operators mapping one Hilbert space to another.

3. THE REPRESENTATION THEOREM

The main result is established in this section and its proof is based on the following:

LEMMA 3.1. *Let* A *be a densely defined and closed linear operator mapping* $\mathscr{H}_1$ *to* $\mathscr{H}_2$. *Then* $(I_2 + AA^*)^{-1}$ *is defined on* $\mathscr{H}_2$ *and maps* $\mathscr{H}_2$ *to* $D(A^*)$. *Moreover,*

$$\| (I_2 + AA^*)^{-1} \| \leq 1 \tag{3.1}$$

and

$$\| A^* (I_2 + AA^*)^{-1} \| \leq 1 . \tag{3.2}$$

If in addition $R(A)$ is closed, then there exists $\gamma > 0$ such that

$$\| (I_2 + AA^*)^{-1} y \| \leq (1/(1+\gamma^2)) \| y \| \qquad \text{for} \quad y \in N(A^*)^{\perp}. \tag{3.3}$$

Proof. The arguments make use of the graph of A^*, $\mathscr{G}(A^*)$; the technique is adopted from [9]. Let $y \in \mathscr{H}_2$ be arbitrary. Since $\mathscr{H}_1 \times \mathscr{H}_2 = \mathscr{G}(A^*) \oplus \mathscr{G}(A^*)^{\perp}$, we can write $(0, y) = (A^* w, w) + (u, v)$ where $(u, v) \in \mathscr{G}(A^*)$. Thus $(-u, v) \in \mathscr{G}(A^{**}) = \mathscr{G}(A)$ and $Au = -v$. Since $y = w + v$, $A^* w = -u$ and we obtain $y = w + AA^* w = (I_2 + AA^*) w$. Thus $(I_2 + AA^*)$ maps $D(A^*)$ onto $\mathscr{H}_2$. Furthermore,

$$\begin{aligned} \| y \|^2 = \| (0, y) \|^2 &= \| (A^* w, w) \|^2 + \| (u, v) \|^2 = \\ &= \| A^* w \|^2 + \| w \|^2 + \| u \|^2 + \| v \|^2. \end{aligned} \tag{3.4}$$

It follows immediately that $(I_2 + AA^*)$ is one-to-one, so $(I_2 + AA^*)^{-1}$ is defined on $\mathscr{H}_2$ and maps into $D(A^*)$. From (3.4) we see that $\| w \| \leq \| y \|$ and $\| A^* w \| \leq \| y \|$; that is, (3.1) and (3.2) hold.

If $R(A)$ is closed, then $R(A^*)$ is closed and there is a constant $\gamma > 0$ such that $\| A^* w \| \geq \gamma \| w \|$ for all $w \in D(A^*) \cap N(A^*)^{\perp}$ and $\| Au \| \geq \gamma \| u \|$ for all $u \in D(A) \cap N(A)^{\perp}$, [4]. For $y \in N(A^*)^{\perp}$, maintaining the above notation, $y = (I_2 + AA^*)^{-1} w$, where $y = w + v = w - Au$. Thus $w \in N(A^*)^{\perp} \cap D(A^*)$ and with $u = -A^* w$,

$$\begin{aligned} \| y \|^2 = \| (I + AA^*) w \|^2 &= \langle w, w \rangle + \langle A^* w, A^* w \rangle + \langle A^* w, A^* w \rangle + \langle Au, Au \rangle \\ &\geq \| w \|^2 + 2\gamma^2 \| w \|^2 + \gamma^2 \| u \|^2 \geq (1 + 2\gamma^2 + \gamma^4) \| w \|^2 . \end{aligned}$$

Hence $\| (I_2 + AA^*)^{-1} y \| \leq \frac{1}{1+\gamma^2} \| y \|$ and (3.3) holds.

THEOREM 3.2. *Let* A *be a densely defined and closed linear operator mapping* $\mathscr{H}_1$ *to* $\mathscr{H}_2$. *If* $y \in D(A^{\dagger})$, *then*

$$A^{\dagger} y = \sum_{k=1}^{\infty} A^* (I_2 + AA^*)^{-k} y . \tag{3.5}$$

Moreover, if $R(A)$ *is closed, then*

$$A^{\dagger} = \sum_{k=1}^{\infty} A^* (I_2 + AA^*)^{-k} \tag{3.6}$$

where the series of bounded operators converges in the uniform operator topology.

Proof. Since, as noted in (2.3), $D(A^{\dagger}) = N(A^*) \oplus R(A)$, for (3.5) it suffices to consider the two cases, $y \in N(A^*)$ and $y \in R(A)$.

If $y \in N(A^*)$, then $y = (I_2 + AA^*) y$. Hence $(I_2 + AA^*)^{-1} y = y$ and we have $\sum_{k=1}^{\infty} A^* (I_2 + AA^*)^{-1} y = 0$. But from (2.4), $N(A^{\dagger}) = N(A^*)$, and we conclude that (3.5) holds for $y \in N(A^*)$.

Now let $y \in R(A)$, $y = Ax$ for $x \in D(A)$. As in the proof of Lemma 3.1, $(I_1 + A^* A)$ maps $D(A)$ onto $\mathscr{H}_1$, so there exists $u \in D(A)$ such that $x = (I_1 + A^*A)\, u$. Thus

$$y = Ax = Au + AA^* Au = (I_2 + AA^*)\, Au\,.$$

This yields, for any $x \in D(A)$,

$$(I_2 + AA^*)^{-1} Ax = A\,(I_1 + A^* A)^{-1} x\,.$$

Using induction we obtain the more general result

$$A^*(I_2 + AA^*)^{-k} Ax = A^*A\,(I_1 + A^*A)^{-k} x \qquad \text{for} \quad k = 1, 2, \cdots. \tag{3.7}$$

Now let $\{E_\lambda\}$ denote the spectral family of the self-adjoint operator $A^* A$, [12]. Then from (3.7),

$$\sum_{k=1}^{n} A^*(I_2 + AA^*)^{-k} y = \sum_{k=1}^{n} \int_0^\infty \frac{\lambda}{(1+\lambda)^k}\, dE_\lambda\, x\,. \tag{3.8}$$

Also since by (2.1), $A^\dagger y = A^\dagger Ax = Px$ and since $\overline{R(A^*)}$ is orthogonal to $N(A) = N(A^* A)$, $A^\dagger y$ is the orthogonal projection of x onto $N(A^*A)^\perp$. Thus if we denote by $\varphi_0(\lambda)$ the function which vanishes for $\lambda = 0$ and otherwise has the value 1, then $A^\dagger y = \int_0^\infty \varphi_0(\lambda)\, dE_\lambda\, x$. Using this with (3.8) we obtain

$$\left\| \sum_{k=1}^{n} A^*(I_2 + AA^*)^{-k} y - A^\dagger y \right\|^2 = \int_0^\infty ((1 - (1+\lambda)^{-n}) - \varphi_0(\lambda))^2\, d(E_\lambda x, x)\,.$$

But $\lim_{n\to\infty} ((1 - (1+\lambda)^{-n}) - \varphi_0(\lambda)) = 0$ for all $\lambda \geq 0$. Hence by the Lebesque dominated convergence theorem it follows that $\sum_{k=1}^{\infty} A^*(I_2 + AA^*)^{-k} y = A^\dagger y$. The proof of (3.5) is complete.

Next suppose that $R(A)$ is closed and consider the uniform convergence of the series in (3.6). Set $B = (I_2 + AA^*)^{-1}$ and $C = A^*(I_2 + AA^*)^{-1}$. From Lemma (3.2), B and C are bounded operators. Since $AC = I_2 - B$, we have

$$\sum_{k=1}^{n} A^*(I_2 + AA^*)^{-k} = \sum_{k=1}^{n} CB^{k-1} = C\left(\sum_{k=1}^{n} B^{k-1}\right). \tag{3.9}$$

From (2.1),

$$C = A^\dagger AC = A^\dagger (I_2 - B)\,. \tag{3.10}$$

Thus, using (3.9) and (3.10),

$$\left\| \sum_{k=1}^{n} A^*(I_2 + AA^*)^{-k} - A^\dagger \right\| = \left\| A^\dagger (I_2 - B)\left(\sum_{k=1}^{n} B^{k-1}\right) A^\dagger \right\| = \| A^\dagger B^n \|\,.$$

By (3.3) of Lemma 3.1, $\| By \| \leq \left(\frac{1}{1+\lambda^2}\right) \| y \|$ for $y \in N(A^*)^{\perp}$, and since B maps $N(A^*)^{\perp}$ to $N(A^*)^{\perp}$, we have $\| B^n y \| \leq \left(\frac{1}{1+\lambda^2}\right)^n \| y \|$. Also if $y \in N(A^*)$, then $By = y$ and by (2.4) $A^{\dagger} By = 0$. Thus we have $\| A^{\dagger} B^n \| \leq \frac{1}{(1+\lambda^2)^n}$ and the uniform convergence of the series (3.6) now follows.

COROLLARY 3.2. *Let* A *be a closed and densely defined operator mapping* $\mathscr{H}_1$ *to* $\mathscr{H}_2$. *Then* $A^{\dagger}$ *is compact if and only if* R (A) *is closed and* $A^*(I_2 + AA^*)^{-1}$ *is compact.*

Proof. If R (A) is closed and $A^* (I_2 + AA^*)^{1-}$ is compact, then from (3.6), we have $A^{\dagger}$ expressed as the limit of compact operators, hence $A^{\dagger}$ is compact.

If $A^{\dagger}$ is compact, then of course $A^{\dagger}$ is continuous and R (A) is closed by (2.8). We see as in the proof of Lemma 3.1 that $AA^* (I_2 + AA^*)^{-1}$ is bounded and it follows that $AA^{\dagger}A^* (I_2 + AA^*)^{-1}$ is compact. However from (3.10) $A^{\dagger} AA^* (I_2 + AA^*)^{-1} = A^* (I_2 + AA^*)^{-1}$.

COROLLARY 3.3. *Let* A *be a closed densely defined linear operator mapping* $\mathscr{H}_1$ *to* $\mathscr{H}_1$. *If for some complex* λ, $(A - \lambda I_1)^{-1}$ *is compact, then* $A^{\dagger}$ *is compact.*

Proof. To show that R (A) is closed, it is sufficient to show that A maps bounded closed sets onto closed sets [4, p. 99]. If S is a bounded closed set and $x_n \in S$ is such that $\{ Ax_n \}$ converges to y, then $\{ Ax_n - \lambda x_n \}$ is a bounded sequence. Hence $x_n = (A - \lambda)^{-1} (A - \lambda) x_n$ has a convergent subsequence and we let x denote its limit. Since A is a closed operator, we have $y = Ax$ and it follows that R (A) is closed.

Since $AA^* (I_1 + AA^*)^{-1}$ and $A^* (I_1 + AA^*)^{-1}$ are bounded operators, the operator $(A - \lambda I_1) A^* (I_1 + AA^*)^{-1}$ is bounded. Multiplying by the compact operator $(A - \lambda I_1)^{-1}$, we see that $A^* (I_1 + AA^*)^{-1}$ is compact and the result follows from Corollary 3.2.

Finally we note that simple direct arguments show that if A is a bounded operator from $\mathscr{H}_1$ to $\mathscr{H}_2$, then $A^{\dagger}$ is compact if and only if R(A) has finite dimension.

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