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Further results for a system of third order differential equations

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Equazioni differenziali ordinarie non lineari. — *Further results for a system of third order differential equations.* Nota di JAMES O. C. EZEILO e HAROON O. TẸJUMỌLA, presentata(*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori in continuazione di alcune loro ricerche precedenti provano due teoremi sulla limitatezza e sull'esistenza di soluzioni periodiche dell'equazione

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

1. This paper is essentially a continuation of the work in a previous paper [2] on the real n-vector equation:

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.1}$$

where A is a constant $n \times n$ matrix, $X \in R_n$ (the real Euclidean n-dimensional space), $G : R_n \to R_n$, $H : R_n \to R_n$ and $P : R \times R_n \times R_n \times R_n \to R_n$, R here being the real line $-\infty < t < \infty$. It will be assumed as basic throughout that $G, H \in C'(R_n)$ and that $P \in C(R \times R_n \times R_n \times R_n)$. We shall examine here two specific properties of solutions of (1.1), namely the strong boundedness property of solutions in which the bounding constant is independent of solutions and the existence of periodic solutions, which were quite known at the time of the earlier paper for the special case $G \equiv B\dot{X}$ (with B a constant $n \times n$ matrix (see [1])) but which we were unable then to extend to (1.1) for more general G.

To achieve the desired extensions to (1.1) we have had to impose some additional restriction on H (see (2.5) and (2.6) below), but we have at the same time seized full advantage of those extra restrictions to dispense with the commutativity condition (in [1] and [2]) on the Jacobian matrices corresponding to G and H and also to relax somewhat the restriction on the eigenvalues of the Jacobian matrix corresponding to G.

2. As in [2] capital letters $A, B, X, Y, Z, \cdots$ with or without suffixes will be used for vectors in R_n as well as for matrices, while Greek letters will be used exclusively for real numbers. The components of a vector will be denoted by the corresponding lower case letters in n places with suffixes running from 1 to n. As usual $J_h(X)$ and $J_g(Y)$ are the Jacobian matrices $(\partial h_i/\partial x_j)$ (and $\partial g_i/\partial y_j$) of the vectors $H(X)$ and $G(Y)$ respectively.

The symbol $\langle X, Y \rangle$ stands for the usual scalar product $\Sigma x_i y_i$ in R_n. The Euclidean length in R_n will be denoted by $\|\cdot\|$.

(*) Nella seduta dell'8 febbraio 1975.

Finally, in what follows, corresponding to any real constant $\delta > 0$ we shall often refer to two functions $\Psi_{h,\delta} : R_n \rightarrow R$, $\lambda_\delta : R_n \rightarrow R^+$ (the set of non negative reals) which are defined for given arbitrary $X \in R_n$ as follows: Let $x_{i_1}, x_{i_2}, \cdots, x_{i_m}$ be those components x_i of X satisfying $|x_i| > \delta$ so that

$$|x_{i_k}| > \delta \quad (k = 1, \cdots, m) \qquad \text{and} \quad |x_i| \leq \delta \qquad (i \neq i_k) .$$

Then

$$\Psi_{h,\delta}(X) \equiv \sum_{k=1}^{m} h_{i_k}(X) \operatorname{sgn} x_{i_k} \quad , \quad \lambda_\delta(X) \equiv \sum_{k=1}^{m} x_{i_k}^2 . \tag{2.1}$$

The main results of the paper are the following two theorems:

THEOREM 1. *Let* $G(O) = O = H(O)$ *and assume that the matrix* A *as well as the Jacobian matrices* $J_g(Y)$ *and* $J_h(X)$ *are symmetric for arbitrary* $X, Y \in R_n$. *Let* δ_α *denote the least eigenvalue of* A *and let* $\lambda_i(J_g(Y))$, $\lambda_i(J_h(X))$ $(i = 1, 2, \cdots, n)$ *denote the eigenvalues of* $J_g(Y)$, $J_h(X)$ *respectively. Suppose also that*

(i) $J_h(X)$ *commutes with* $J_h(X')$ *for arbitrary* $X, X' \in R_n$

(ii) $\delta_\alpha > 0$ *and there are constants* $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ *with*

$$\delta_1 \delta_\alpha > \delta_2 \tag{2.2}$$

such that

$$\lambda_i(J_g(Y)) \geq \delta_1 \qquad (\|Y\| \geq \rho > 0) \tag{2.3}$$

$$\delta_2 \geq \lambda_i(J_h(X)) \geq \delta_3 \qquad \text{for all} \quad X \in R_n . \tag{2.4}$$

(iii) *Each component* $h_i(X)$ $(i = 1, 2, \cdots, n)$ *of* $H(X)$ *satisfies*

$$h_i(X) \operatorname{sgn} x_i \geq -\Delta \tag{2.5}$$

for some constant $\Delta \geq 0$. *Furthermore*

$$\Psi_{h,\delta}(X) \rightarrow \infty \qquad \text{as} \quad \lambda_\delta(X) \rightarrow \infty \tag{2.6}$$

for any fixed constant $\delta > 0$

(iv) *the vector* P *satisfies*

$$\|P(t, X, Y, Z)\| \leq \Delta_0 + \Delta_1 \|Y\| + \|Z\|) \tag{2.7}$$

for all t considered and for arbitrary $X, Y, Z \in R_n$, *where* Δ_0, Δ_1 *are constants.*

Then there exist constants $\Delta_2 > 0$, $\varepsilon > 0$ *whose magnitudes depend only on* Δ_0, A, G *and* H *such that if* $\Delta_1 \leq \varepsilon$ *then every solution* $X(t)$ *of* (1.1) *ultimately satisfies*

$$\|X(t)\| \leq \Delta_2 \quad , \quad \|\dot{X}(t)\| \leq \Delta_2 \quad , \quad \|\ddot{X}(t)\| \leq \Delta_2 . \tag{2.8}$$

THEOREM 2. *Suppose further to the conditions of Theorem 1 that there exists* $\omega > 0$ *such that*

$$P(t+\omega, X, Y, Z) = P(t, X, Y, Z)$$

for all t, X, Y, Z. *Then if* $\Delta_1 \leq \varepsilon$, *with* ε *as determined in Theorem 1, there exists at least one periodic solution of* (1.1) *with period* ω.

Note that in the case when $h_i(X) \equiv h_i(x_i)$ (so that $\partial h_i/\partial x_j = 0$ $(i \neq j)$) the conditions (2.5) and (2.6) are automatically implied by (2.4).

3. In what follows each Greek letter δ (with or without a suffix), which appears, stands for a positive constant whose magnitude depends only on A, G, H and Δ_0. The δ's without subscripts are not necessarily the same each time they occur, but the δ's with suffixes attached: $\delta_0, \delta_1, \delta_2, \cdots$ retain their identities throughout.

4. PROOF OF THEOREM 1

It is convenient to consider (1.1) in the system form:

$$\dot{X} = Y \quad , \quad \dot{Y} = Z \quad , \quad \dot{Z} = -AZ - G(Y) - H(X) + P(t, X, Y, Z) \tag{4.1}$$

and to prove then that every solution $(X(t), Y(t), Z(t))$ of (4.1) ultimately satisfies

$$\| X(t) \| \leq \delta \quad , \quad \| Y(t) \| \leq \delta \quad , \quad \| Z(t) \| \leq \delta , \tag{4.2}$$

if the constant Δ_1 in (2.7) is sufficiently small.

Now let $\beta_1 > 0$ be a constant fixed, as is possible in view of (2.2), such that

$$\delta_1 \delta_2^{-1} > \beta_1 > \delta_\alpha^{-1} , \tag{4.3}$$

and define the real continuous function $\psi(\alpha, \beta)$, for arbitrary $\alpha, \beta \in R$, by

$$\psi(\alpha, \beta) = \begin{cases} \alpha \operatorname{sgn} \beta , & \text{if } |\beta| \geq |\alpha| , \\ \beta \operatorname{sgn} \alpha , & \text{if } |\alpha| \geq |\beta| . \end{cases} \tag{4.4}$$

Our main tool in the proof of (4.2) is the continuous function

$$V : R_n \times R_n \times R_n \to R$$

given for arbitrary $X, Y, Z \in R_n$ by

$$V(X, Y, Z) = V_1 + V_2$$

where

$$\begin{aligned} 2V_1 = 2\int_0^1 \langle H(\sigma X), X\rangle d\sigma + 2\beta_1 \int_0^1 \langle G(\sigma Y), Y\rangle d\sigma + \\ + \beta_1 \| Z \|^2 + 2\langle Y, Z\rangle + 2\beta_1 \langle Y, H(X)\rangle + \langle AY, Y\rangle , \end{aligned} \tag{4.5}$$

$$V_2 = \sum_{i=1}^{n} \psi_i , \qquad \psi_i \equiv \psi(x_i, z_i) .$$

The function V_1 is essentially the same as the function Ψ on p. 408 of [2]. We shall show that, subject to the given conditions on A, G and H,

$$(4.6) \qquad V \to +\infty \quad \text{as} \quad \| X \|^2 + \| Y \|^2 + \| Z \|^2 \to \infty$$

and also that there exists δ_3 such that, corresponding to any solution $(X(t), Y(t), Z(t))$ of (4.1), the limit $\dot{V}(t)$ defined by

$$\dot{V}^*(t) \equiv \limsup_{\nu \to +0} \frac{V(X(t+\nu), Y(t+\nu), Z(t+\nu)) - V(X(t), Y(t), Z(t))}{\nu}$$

satisfies

$$(4.7) \qquad \dot{V}^*(t) \leq -1 \quad \text{if} \quad \| X(t) \|^2 + \| Y(t) \|^2 + \| Z(t) \|^2 \geq \delta_3 ,$$

provided further that P satisfies (2.7) with Δ_1 sufficiently small. According to a well known Yoshizawa technique for investigating the boundedness of solutions of systems of differential equations, these two results (4.6) and (4.7) are quite sufficient for (4.2), so that the theorem will follow once (4.6) and (4.7) are established.

Coming now to (4.6) note first that $2V_1$ can be reset in the form:

$$(4.8) \qquad \begin{aligned} 2V = {} & \beta_1 \| Z + \beta_1^{-1} Y \|^2 + \langle (A - \beta_1^{-1} I) Y, Y \rangle + \\ & + \beta_1 \delta_1^{-1} \| H(X) + \delta_1 Y \|^2 + R_1(X) + \beta_1 R_2(Y) \end{aligned}$$

where I is the identity $n \times n$ matrix and

$$R_1(X) \equiv 2 \int_0^1 \langle H(\sigma X), X \rangle \, d\sigma - \beta_1 \langle H(X), \delta_1^{-1} H(X) \rangle ,$$

$$R_2(Y) \equiv 2 \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma - \delta_1 \| Y \|^2 .$$

Next, as shown on p. 292 of [1],

$$(A - \beta_1^{-1} I) Y, Y \geq (\delta_a - \beta_1^{-1}) \| Y \|^2 \geq 0$$

since $\delta_a - \beta_1^{-1} > 0$, by (4.3). Also

$$R_1(X) \geq \delta \| X \|^2$$

as is readily checked by following through the estimate of 2.3 (13) of [1] and using the fact that the eigenvalues $\lambda_i(J_h(X))$ here are now subject to (2.4) for *all* X. Finally, by rewriting $R_2(Y)$ in the form

$$R_2(Y) = 2 \int_0^1 \int_0^1 \langle \{ J_g(\sigma_1 \sigma_2 Y) - \delta_1 I \} Y, Y \rangle \, \sigma_1 \, d\sigma_1 \, d\sigma_2$$

and then breaking up this double integral for $R_2(Y)$ into four component parts (in the same way as in our treatment of 2.3 (13) on p. 293 of [1]) in each of the two distinct cases: $\| Y \| \geq \rho$, $\| Y \| \leq \rho$, it can be verified that

$$R_2(Y) \geq -\delta(1 + \| Y \|).$$

By combining these various estimates with (4.8) we obtain that

$$2V_1 \geq \beta_1 \| Z + \beta_1^{-1} Y \|^2 + \delta(\| X \|^2 + \| Y \|^2 - \| Y \| - 1).$$

But, by (4.4) and (4.5),

$$| V_2 | \leq \sum_{i=1}^{n} | x_i | \leq \delta \| X \|.$$

Hence

$$V \geq \frac{1}{2} \beta_1 \| Z + \beta_1^{-1} Y \|^2 + \delta(\| X \|^2 + \| Y \|^2 - \| X \| - \| Y \| - 1)$$

from which (4.6) follows at once.

Turning now to (4.7), let $(X, Y, Z) \equiv (X(t), Y(t), Z(t))$ be any solution of (4.1). Then as shown in [2] (see specially 3.2 (11))

$$\dot{V}_1 = -\theta(X, Y) - \langle(\beta_1 A - I) Z, Z\rangle + \langle Y + \beta_1 Z, P(t, X, Y, Z)\rangle, \tag{4.9}$$

where

$$\theta(X, Y) = \langle G(Y), Y\rangle - \beta_1 \langle J_h(X) Y, Y\rangle. \tag{4.10}$$

It is also clear from (4.4) that

$$\dot{\psi}_i^*(t) = \begin{cases} y_i \operatorname{sgn} z_i, & \text{if } | z_i | \geq | x_i | \\ -\left(\sum_{j=1}^{n} a_{ij} z_j + g_i(Y) + h_i(X) - p_i(t, X, Y, Z)\right) \operatorname{sgn} x_i, & \text{if } | x_i | \geq | z_i | \end{cases}$$

for $i = 1, 2, \cdots, n$, where a_{ij} is the $(i, j)^{\text{-th}}$ element of A. Thus if $z_{i_1}, z_{i_2}, \cdots, z_{i_m}$ denote the z_i's out of the set $(z_1(t), z_2(t), \cdots, z_n(t))$ which satisfy $| z_i | < | x_i |$, so that

$$| z_{i_k} | < | x_{i_k} | \quad (k = 1, \cdots, m) \quad \text{and} \quad | z_i | \geq | x_i | \quad (i \neq i_k) \tag{4.11}$$

then

$$\dot{V}_2^*(t) = \sum_{i \neq i_k} y_i \operatorname{sgn} z_i - \tag{4.12}$$

$$- \sum_{k=1}^{m} \left(\sum_{j=1}^{n} a_{i_k j} z_j + g_{i_k}(Y) + h_{i_k}(X) - p_{i_k}(t, X, Y, Z)\right) \operatorname{sgn} x_{i_k}.$$

Hence, by (2.5) and (2.7)

$$\dot{V}_2^* \leq \delta_4 (\| Y \| + \| Z \| + \| G(Y) \| + \Delta_1 (\| Y \|^2 + \| Y \| \cdot \| Z \| + 1) \tag{4.13}$$

always. Thus, since

$$\langle(\beta_1 A - I) Z , Z\rangle = \beta_1 \langle(A - \beta_1^{-1} I) Z , Z\rangle$$
$$\geq \beta_1 (\delta_a - \beta_1^{-1}) \| Z \|^2 \geq 0 ,$$

as we have seen earlier on, we have from (4.9), (4.10) and (4.13) that

$$\dot{V}^* \leq - \delta_5 \| Z \|^2 + \delta_4 \Delta_1 (\| Y \|^2 + \| Y \| \cdot \| Z \|) + + \delta (\| Y \| + \| Z \| + 1) - \varphi (X , Y) , \tag{4.14}$$

where

$$\varphi (X , Y) \equiv \langle G (Y) , Y\rangle - \beta_1 \langle J_h (X) Y , Y\rangle - \delta_4 \| G (Y) \| .$$

If we now set

$$\delta_6 = \frac{1}{2} (1 - \delta_1^{-1} \delta_2 \beta_1)$$

and note, by the way, that $\delta_6 > 0$ since $\delta_1^{-1} \delta_2 \beta_1 < 1$ (by (4.3)), then φ may be recast in the form:

$$\varphi = \delta_6 \varphi_1 + \delta_6 \varphi_2 + \beta_1 \varphi_3$$

where

$$\varphi_1 \equiv \langle G (Y) , Y\rangle ,$$
$$\varphi_2 \equiv \langle G (Y) , Y\rangle - \delta_4 \delta_5^{-1} \| G (Y) \| ,$$
$$\varphi_3 = \delta_1^{-1} \delta_2 \langle G (Y) , Y\rangle - \langle J_h (X) Y , Y\rangle .$$

It is clear from (2.4) that

$$\langle J_h (X) Y , Y\rangle \leq \delta_2 \| Y \|^2 . \tag{4.15}$$

Also, resulting from (2.3), we have that

$$\langle G (Y) , Y\rangle \geq \delta_1 \| Y \|^2 - \delta \| Y \| - \delta \tag{4.16}$$

from which it follows, in turn, by means of an obvious substitution that

$$\langle G (Y) - G (Q) , Y - Q\rangle \geq \delta_1 \| Y - Q \|^2 - \delta \| Y - Q \| - \delta \tag{4.17}$$

for arbitrary $Y , Q \in R_n$. The inequalities (4.15) and (4.16) show at once that

$$\varphi_3 \geq - \delta \| Y \| - \delta .$$

Also, since φ_2 may, on setting

$$Q \equiv \delta_4 \delta_5^{-1} (\operatorname{sgn} g_1 , \operatorname{sgn} g_2 , \cdots , \operatorname{sgn} g_n) ,$$

be put in the form

$$\varphi_2 = \langle G (Y) , Y\rangle - \langle G (Y) , Q\rangle$$
$$\equiv \langle G (Y) - G (Q) , Y - Q\rangle + \langle G (Q) , Y\rangle - \langle G (Q) , Q\rangle ,$$

it is clear from (4.17) that

$$\varphi_2 \geq \delta_1 \| Y - Q \|^2 - \delta \| Y - Q \| - \delta \| Y \| - \delta$$
$$\geq - \delta \| Y \| - \delta$$

for all Y. Hence

$$\varphi \geq \delta_6 \| Y \|^2 - \delta \| Y \| - \delta$$

and therefore, by (4.14)

$$\dot{V}^* \leq - \delta_7 (\| Y \|^2 + \| Z \|^2) + \delta_4 \Delta_1 (\| Y \|^2 + \| Y \| \cdot \| Z \|) +$$
$$+ \delta (\| Y \| + \| Z \| + 1) .$$

Thus if, for example,

$$\Delta_1 \leq \frac{1}{3} \delta_7 \delta_4^{-1} , \tag{4.18}$$

which we suppose henceforth, then

$$\dot{V}^*_{(t)} \leq - \frac{1}{2} \delta_7 (\| Z \|^2 + \| Y \|^2) + \delta (\| Y \| + \| Z \| + 1) \tag{4.19}$$
$$\leq - 1$$

if $\| Y(t) \|^2 + \| Z(t) \|^2 \geq \delta_8^2$, for δ_8 sufficiently large.

Suppose however that $\| Y(t) \|^2 + \| Z(t) \|^2 < \delta_8^2$ but with $X(t)$ such that $\| X(t) \| > \delta_8$. Then clearly $| z_i | < | x_i |$ for some i, and let $z_{i_1}, z_{i_2}, \cdots, z_{i_m}$ be those z_i's satisfying $| z_i | < | x_i |$, so that the situation with respect to the z_i's and x_i's is as in (4.11). Then, always recalling that $\| Y(t) \|^2 + \| Z(t) \|^2 < \delta_8^2$ here, we will have from (2.1), (4.9) and (4.12) that

$$\dot{V}^*(t) \leq \delta - \Psi_{h,\delta_8}(X) \tag{4.20}$$
$$\leq - 1 ,$$

by (2.6), provided that $\lambda_{\delta_8}(X) \geq \delta_9^2$, with δ_9 sufficiently large. But, by our definition (2.1) of $\lambda_\delta(X)$,

$$\| X \|^2 = \lambda_{\delta_8}(X) + \sum_{i \neq i_k} x_i^2$$

and by (4.11), $\sum_{i \neq i_k} x_i^2 \leq \sum_{i \neq i_k} z_i^2 < (n - m) \delta_8^2$ since $\| Z(t) \| < \delta_8$. Thus (4.20) implies that

$$\dot{V}^* \leq - 1 \quad \text{if} \quad \| Y \|^2 + \| Z \|^2 < \delta_8^2 \quad \text{but} \quad \| X \| \geq \delta_{10} \tag{4.21}$$

where $\delta_{10}^2 = \delta_9^2 + (n - m) \delta_8^2$.

The results (4.19) and (4.20) show that

$$\dot{V}^* \leq - 1 \quad \text{if} \quad \| X \|^2 + \| Y \|^2 + \| Z \|^2 \geq \delta_8^2 + \delta_{10}^2$$

which establishes (4.7), and thus concludes our verification of Theorem 1, with $\varepsilon = \frac{1}{3} \delta_7 \delta_4^{-1}$ (see (4.18)).

5. PROOF OF THEOREM 2

The proof is by the Leray-Schauder fixed point technique and the procedure is almost exactly as in § 4 of [1] with $B\dot{X}$ replaced by $G(\dot{X})$, so that full details will be omitted here.

A convenient starting point will be the parameter (μ) dependent equation:

$$(5.1)\qquad \dddot{X} + A\ddot{X} + \{(1-\mu)\,\delta_1\dot{X} + \mu G(\dot{X})\} + \{(1-\mu)\,\delta_2 X + \mu H(x)\} = \\ = \mu P(t, X, \dot{X}, \ddot{X})$$

(with $0 \leq \mu \leq 1$), which reduces to the linear system

$$(5.2)\qquad \dddot{X} + A\ddot{X} + \delta_1\dot{X} + \delta_2 X = 0$$

when $\mu = 0$ and to the original equation (1.1) when $\mu = 1$. It is clear from Corollary 1 to Theorem 1 of [1] that the trivial soluzion $X \equiv 0$ of (5.2) is asymptotically stable in the large. Thus the arguments in § 4 of [1] will be applicable, and the existence of an ω-periodic solution of (1.1) will therefore follow, if it can be shown, just as in [1], that there exists a constant δ, *independent of* μ $(0 \leq \mu \leq 1)$ such that every solution $X(t)$ of (5.1) ultimately satisfies

$$(5.3)\qquad \|X(t)\| \leq \delta \quad , \quad \|\dot{X}(t)\| \leq \delta \quad , \quad \|\ddot{X}(t)\| \leq \delta\,.$$

To establish (5.3) it is convenient as before to consider (5.1) in the system form:

$$(5.4)\qquad \dot{X} = Y\ ,\ \dot{Y} = Z\ ,\ \dot{Z} = -AZ - G^*(Y) - H^*(X) + \mu P(t, X, Y, Z)$$

where

$$G^*(Y) = (1-\mu)\,\delta_1 Y + \mu G(Y)$$
$$H^*(X) = (1-\mu)\,\delta_2 X + \mu H(X)$$

and to prove then, in place of (5.3), that every solution $(X(t), Y(t), Z(t))$ of (5.4) ultimately satisfies

$$(5.5)\qquad \|X(t)\| \leq \delta \quad , \quad \|Y(t)\| \leq \delta \quad , \quad \|Z(t)\| \leq \delta\,.$$

Our procedure for this is by the Yoshizawa technique as explained in § 4 using the same V as before but with G^*, H^* replacing G, H respectively. The details will follow the exact outline in § 4 and we will therefore not elaborate further except only to verify that the functions G^* and H^* satisfy precisely the same conditions with respect to (5.4) as G and H respectively with respect to (4.1).

It is clear from the definition of H^* that

$$J_{h^*}(X) = (1-\mu)\,\delta_2 I + \mu J_h(X)$$

so that $J_{h^*}(X)$, $J_{h^*}(X^1)$ necessarily commute if $J_h(X)$ and $J_h(X^1)$ commute.

Also since

$$\lambda_i(J_{h^*}(X)) = \mu\lambda_i(J_h(X)) + (1-\mu)\,\delta_2$$

$$\lambda_i(J_{g^*}(Y)) = \mu\lambda_i(J_g(Y)) + (1-\mu)\,\delta_1$$

we have that

$$\lambda_i(J_{g^*}(Y)) \geq \delta_1 \qquad \text{for} \quad \|Y\| \geq \rho \qquad \text{(by (2.3))}$$

and that

$$\delta_2 \geq \lambda_i(J_{h^*}(X)) \geq \delta_3^* \qquad \text{for all} \quad X\,,$$

by (2.4), where $\delta_3^* = \frac{1}{2}\min(\delta_2\,,\,\delta_3) > 0$, so that the conditions of hypothesis (ii) of Theorem 1 are also valid for G^* and H^*. Next

$$\begin{aligned} h_i^*(X)\,\operatorname{sgn} x_i &= (1-\mu)\,\delta_2\,|x_i| + \mu h_i(X)\,\operatorname{sgn} x_i \\ &> -\Delta \end{aligned}$$

for arbitrary X, by (2.5). Finally, since

$$\Psi_{h^*,\delta}(X) = \mu\Psi_{h,\delta}(X) + (1-\mu)\,\delta_2 \sum_{k=1}^{m} |x_{i_k}|$$

it is clear from (2.6) that for any given constant δ_{11} there exists δ_{12} independent of μ $(0 \leq \mu \leq 1)$ such that

$$\Psi_{h^*,\delta}(X) \geq \delta_{11} \qquad \text{for} \quad \lambda_\delta(X) \geq \delta_{12}\,.$$

Thus H^* also satisfies the condition (iii) of Theorem 1 and the verification of (5.5) can therefore follow as indicated earlier.

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