Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Haroon O. Tejumola

On a certain nonlinear fourth order ordinary differential equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.2, p. 137–142.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_2_137_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Equazioni differenziali ordinarie non lineari. — *On a certain non-linear fourth order ordinary differential equation.* Nota di HAROON O. TẸJUMỌLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dànno due criteri sufficienti per la limitatezza delle soluzioni dell'equazione

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + \varphi(x)\dot{x} + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

dove a_1, a_2, a_4 sono costanti $\varphi(x), p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ funzioni continue dei loro argomenti e nel caso che p sia funzione periodica di t si dà anche un criterio sufficiente per l'esistenza di almeno una soluzione periodica.

1. We shall be concerned here with the differential equation

$$x^{\text{iv}} + a_1\dddot{x} + a_2\ddot{x} + \varphi(x)\dot{x} + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{1.1}$$

where a_1, a_2, a_4 are constants and φ and p are continuous functions of the arguments shown in (1.1).

In the case $p \equiv 0$ in (1.1), Ogurcov [1] showed that if

$$a_1 > 0 \quad , \quad a_2 > 0 \quad , \quad a_4 > 0 \tag{1.2}$$

and if

$$\varphi(x) > 0 \quad \text{and} \quad \theta(x) \equiv a_1 a_2 \varphi(x) - \varphi^2(x) - a_1^2 a_4 > 0 \tag{1.3}$$

for all x, then every solution $x(t)$ satisfies

$$x(t) \to 0 \quad , \quad \dot{x}(t) \to 0 \quad , \quad \ddot{x}(t) \to 0 \quad , \quad \dddot{x}(t) \to 0 \quad \text{as} \quad t \to +\infty . \tag{1.4}$$

Subsequently, Ezeilo and Tẹjumọla [2] considered the case p not necessarily identically zero in (1.1) and showed that every solution is ultimately bounded (with bounding constant independent of solutions) if, in addition to (1.2), $|p(t, x, y, z, u)|$ is bounded for all t, x, y, z, u and if

$$\varphi(x) > 0 \quad \text{and} \quad \theta(x) \geq \delta \quad (\delta > 0 \text{ a constant}) \quad \text{for} \quad |x| \geq 1. \tag{1.5}$$

Let Φ and Ψ be the functions defined by

$$\Phi(x) = \int_0^x \varphi(s)\, ds \quad , \quad \Psi(x) = a_1 a_2 x\, \Phi(x) - \Phi^2(x) - a_1^2 a_4 x^2. \tag{1.6}$$

Then

$$\Psi(x) = x \int_0^x \theta(s)\, ds + \left\{ x \int_0^x \varphi^2(s)\, ds - \Phi^2(x) \right\},$$

(*) Nella seduta dell'8 febbraio 1975.

where $\left\{x\int_0^x \varphi^2(s)\,ds - \Phi^2(x)\right\} \geq 0$ for all x, as can be verified by the use of Schwartz's inequality. Thus, the condition (1.3) implies that

$$x^{-1}\,\Phi(x) > 0 \quad \text{and} \quad x^{-2}\,\Psi(x) > 0 \qquad (x \neq 0), \tag{1.7}$$

while (1.5) implies that

$$x^{-1}\,\Phi(x) > 0 \quad \text{and} \quad x^{-2}\,\Psi(x) \geq \delta \qquad \text{for} \quad |x| \geq 1. \tag{1.8}$$

The main object of this note is to point out that the results [1] and [2] remain valid if (1.3) and (1.5) are replaced by the weaker conditions (1.7) and (1.8). In the case when p is periodic in t, it will be shown that (1.8) is also sufficient for the existence of periodic solutions of (1.1).

In what follows, let Φ and Ψ be defined as in (1.6). Our first result concerns the case $p \equiv 0$ and is as follows.

THEOREM 1. *Given the equation*

$$x^{\text{iv}} + a_1\dddot{x} + a_2\ddot{x} + \varphi(x)\dot{x} + a_4 x = 0 \tag{1.9}$$

suppose that conditions (1.2) *and* (1.7) *hold. Then every solution* $x(t)$ *of* (1.9) *satisfies* (1.4).

For the general equation (1.1) we prove the following:

THEOREM 2. *Let conditions* (1.2) *and* (1.8) *hold and suppose that* p *satisfies*

$$|p(t, x, y, z, u)| \leq A \qquad \textit{for all} \quad t, x, y, z \textit{ and } u, \tag{1.10}$$

where A *is a finite constant. Then there exists a constant* K *whose magnitude depends only on* a_1, a_2, a_4, δ, A *and* φ *such that every solution* $x(t)$ *of* (1.1) *ultimately satisfies*

$$|x(t)| \leq K \quad , \quad |\dot{x}(t)| \leq K \quad , \quad |\ddot{x}(t)| \leq K \quad , \quad |\dddot{x}(t)| \leq K .$$

THEOREM 3. *Suppose, further to the conditions of Theorem* 2, *that* p *satisfies*

$$p(t+\omega, x, y, z, u) = p(t, x, y, z, u) \qquad \textit{for all} \quad t, x, y, z, u .$$

Then equation (1.1) *admits of at least one* ω*-periodic solution.*

Note that in the special case $\varphi(x) \equiv a_3$, $a_3 > 0$ a constant, the condition (1.7) (and, indeed (1.3)) together with (1.2) reduces to the Routh-Hurwitz stability criteria

$$a_i > 0 \quad (i = 1, 2, 3, 4), \qquad (a_1 a_2 - a_3)\, a_3 - a_1^2 a_4 > 0 \tag{1.11}$$

for (1.9) with $\varphi(x) \equiv a_3$.

In what follows, we shall adopt the notations in [2] and use the letters D_i, $i = 1, 2, 3, \cdots$ to denote finite positive constants whose magnitudes depend only on the constants $a_1, a_2, a_3, a_4, \delta$, A as well as on the function φ, but are independent of solutions of any differential equation under consideration. In § 4, the D_i's are, in addition, independent of the parameter μ employed in defining equation (4.1).

2. Proof of Theorem 1

The differential equation (1.9) is equivalent to the system

$$(2.1)\qquad \dot{x}=y \ , \quad \dot{y}=z \ , \quad \dot{z}=u-a_1 z-\Phi(x) \ , \quad \dot{u}=-a_2 z-a_4 x \ .$$

Consider Ogurcov's function [1] $V_1 = V_1(x, y, z, u)$ defined here by

$$(2.2)\qquad V_1 = Q + a_1 \int_0^x \{\Phi(s) - a_1 a_2^{-1} a_4 s\} \, ds \ ,$$

where

$$2Q = (a_2^2 - 2a_4 + a_1^2 a_2^{-1} a_4) x^2 + (a_1^2 + a_2^3 a_4^{-1} - 3a_2) y^2 + 2z^2 + a_2 a_4^{-1} u^2 + \\ + 2a_2 xz - 2a_1 xu + 2a_1 yz + 2(a_2^2 a_4^{-1} - 2) yu$$

is a positive definite quadratic form (for proof, see [1]). Since the integral in (2.2) is, by (1.7), non-negative, the function V_1 satisfies

$$V_1(x, y, z, u) \to +\infty \qquad \text{as} \quad x^2 + y^2 + z^2 + u^2 \to \infty \ .$$

Let $(x, y, z, u) \equiv (x(t), y(t), z(t), u(t))$ be any solution of (2.1). It is easy to verify from (2.2) and (2.1) that

$$(2.4)\qquad \dot{V}_1 = -a_1^{-1}[\Phi(x) + a_1 z]^2 - a_1^{-1}\Psi(x) \ ,$$

so that, in view of (1.7), $\dot{V}_1 \leq 0$. Thus all solutions of (2.1) are bounded for all $t \geq 0$. Let the solution (x, y, z, u) of (2.1) satisfy the condition $\dot{V}_1 \equiv 0$. Then, from (2.4),

$$\Psi(x) \equiv 0 \qquad \text{and} \qquad [\Phi(x) + a_1 z] \equiv 0$$

and, by (1.6) and (1.7), this implies that

$$x \equiv 0 \quad , \quad z \equiv 0 \ .$$

It therefore follows from system (2.1) that

$$y \equiv 0 \qquad \text{and} \qquad u \equiv 0 \ .$$

Theorem 1 now follows in view of [4; Theorem 1.11].

3. Proof of Theorem 2

The procedure is the same as for the result [2], and we shall merely indicate the modifications necessary in the arguments in [2].

Take (1.1) in the system form

$$(3.1)\quad \dot{x}=y \ , \ \dot{y}=z \ , \ \dot{z}=u-a_1 z-\Phi(x) \ , \ \dot{u}=-a_2 z-a_4 x+p(t,x,y,z,u)$$

and consider the function (same as (4.3) of [2]) $V = V(x, y, z, u)$ defined here by

$$V = V_1 + V_2 + V_3, \tag{3.2}$$

where V_1 is the function (2.2) and

$$\begin{cases} V_2 = \begin{pmatrix} -\lambda x \operatorname{sgn} y, & \text{if } |y| \geq |x| \\ -\lambda y \operatorname{sgn} x, & \text{if } |x| \geq |y| \end{pmatrix} \\ V_3 = \begin{pmatrix} -\lambda z \operatorname{sgn} u, & \text{if } |u| \geq |z| \\ -\lambda u \operatorname{sgn} z, & \text{if } |z| \geq |u| \end{pmatrix} \end{cases} \tag{3.3}$$

$\lambda > 0$ being, as yet, an arbitrary constant. Observe that (1.8) implies that $x^{-1}\Phi(x) > a_2^{-1}(a_1 a_4 + a_1^{-1}\delta)$ if $|x| \geq 1$, so that by the continuity of Φ

$$\int_0^x \{\Phi(s) - a_1 a_2^{-1} a_4 s\}\, ds \geq -D_1 \qquad \text{for all } x .$$

Thus

$$V \geq Q - \lambda(|y| + |u|) - D_1$$

(since $|V_2| \leq \lambda |y|$ and $|V_3| \leq \lambda |u|$) and hence

$$V(x, y, z, u) \to +\infty \qquad \text{as } x^2 + y^2 + z^2 + u^2 \to \infty, \tag{3.4}$$

since Q is a positive definite quadratic form.

For any solution $(x, y, z, u) \equiv (x(t), y(t), z(t), u(t))$ of (3.1), we have, by a straightforward differentiation of (2.2), that

$$\dot{V}_1 = -a_1^{-1}[\Phi(x) + a_1 z]^2 - a_1^{-1}\Psi(x) + \{a_2 a_4^{-1} u + (a_2^2 a_4^{-1} - 2) y - a_1 x\} p. \tag{3.5}$$

Now choose D_2 such that

$$D_2 > (a_1^2 a_4 + \delta) + \max_{|x| \leq 1} |a_1 a_2 x \Phi(x) - \Phi^2(x)| .$$

Then

$$\Psi(x) \geq \delta x^2 - D_2 \qquad \text{for all } x, \tag{3.6}$$

as can be easily verified from (1.6) and (1.8). Observe next from (1.8) that $0 < x^{-1}\Phi(x) < a_1 a_2$ if $|x| \geq 1$, so that by the continuity of Φ,

$$|\Phi(x)| \leq a_1 a_2 |x| + D_3 \qquad \text{for all } x$$

with D_3 sufficiently large. Thus

$$\Phi^2(x) \leq a_1 a_2 D_4 x^2 + D_2 D_4 \quad (D_4 \equiv a_1 a_2 + D_2) \qquad \text{for all } x,$$

and hence

$$(3.7)\qquad a_1^{-1}[\Phi(x)+a_1 z]^2+\tfrac{1}{2}a_1^{-1}\delta x^2\geq a_1^{-1}[\Phi(x)+a_1 z]^2+ \\ +\tfrac{1}{2}\delta(a_1^2 a_2 D_4)^{-1}\Phi^2(x)-\tfrac{1}{2}\delta(a_1^2 a_2)^{-1}D_2 \\ \geq D_5\{z^2+\Phi^2(x)\}-D_6$$

for a sufficiently small D_5 and with $D_6=\tfrac{1}{2}\delta(a_1^2 a_2)^{-1}D_2$. On using the estimate (3.7) and (3.6) in (3.5) and noting from (1.10) that $|p|\leq A$, it follows that there are constants D_7 and D_8 such that

$$\dot{V}_1\leq -D_7(x^2+z^2)+D_8(1+|x|+|y|+|u|).$$

From this point onwards, the arguments in [2; § 7] apply here. By defining $\dot{V}^*=\dot{V}_1+\dot{V}_2^*+\dot{V}_2^*$ as in [2; Lemma 2] but relative to the system (3.1) and noting that the estimates (7.2) and (7.3) of [2] hold respectively for $\dot{V}_2^*$ and $\dot{V}_2^*$ with $\lambda=2D_8$, it can be shown, just as in [2, § 7], that

$$\dot{V}^*\leq -1 \quad \text{if} \quad x^2+y^2+z^2+u^2\geq D_9 .$$

This, together with (3.4), implies Theorem 2 in the usual way.

4. Proof of Theorem 3

The proof here is by Schaefer's Theorem [3; § 5]. Let $a_3>0$ be a constant satisfying (1.11) and consider the parameter μ-dependent equation

$$(4.1)\qquad \begin{cases} x^{iv}+a_1\dddot{x}+a_2\ddot{x}+\varphi_\mu(x)\dot{x}+a_4 x=\mu p(t,x,\dot{x},\ddot{x},\dddot{x}) \qquad (0\leq\mu\leq 1), \\ \varphi_\mu(x)\equiv(1-\mu)a_3+\mu\varphi(x), \end{cases}$$

which reduces to the original equation (1.1) for the value $\mu=1$. The equation (4.1) itself is equivalent to the system

$$(4.2)\qquad \dot{x}=y \;,\; \dot{y}=z \;,\; \dot{z}=u-a_1 z-\Phi_\mu(x) \;,\; \dot{u}=-a_2 z-a_4 x+\mu p$$

$$\Phi_\mu(x)\equiv\int_0^x \varphi_\mu(s)\,ds\,,$$

and, in view of the choice of a_3, we may use the system (4.2) to set up the operator required in the application of the Schaefer's Theorem (see [3; § 5]). Indeed, following the arguments in [3], it suffices here to show that solutions of (4.2) are ultimately bounded, with the bounding constant independent of solutions and of μ.

Let $V=V_1+V_2+V_3$ be the function (3.2) but with $\Phi_\mu(x)$ in place of $\Phi(x)$. That is,

$$(4.3)\qquad V_1=Q+a_1\int_0^x\{\Phi_\mu(s)-a_1 a_2^{-1}a_4 s\}\,ds\,,$$

where Q is given by (2.3). Then, by the definition of $\Phi_\mu(x)$,

$$2\int_0^x \{\Phi_\mu(s) - a_1 a_2^{-1} a_4 s\}\, ds = 2\mu \int_0^x \{\Phi(s) - a_1 a_2^{-1} a_4 s\}\, ds + $$
$$+ (1-\mu)(a_3 - a_1 a_2^{-1} a_4)\, x^2 .$$

Since the integral on the right hand side is, by (1.8), non-negative if $|x| \geq 1$, and $(a_3 - a_1 a_2^{-1} a_4) > 0$ by (1.11), it follows from the continuity of Φ that

$$2\int_0^x \{\Phi_\mu(s) - a_1 a_2^{-1} a_4 s\}\, ds \geq -D_{10} \qquad \text{for all} \quad x .$$

Thus, as in the preceding case, V satisfies (3.4) uniformly in μ, since Q is positive definite in x, y, z, and u.

Let $(x, y, z, u) \equiv (x(t), y(t), z(t), u(t))$ be any solution of (4.2). It is easy to verify from (4.3) that

$$\dot{V}_1 = -W + \mu\{a_2 a_4^{-1} u - (a_2^2 a_4^{-1} - 2)\, y - a_1 x\}\, p ,$$

where
$$W \equiv a_1 z + 2z\,\Phi_\mu(x) + a_2 x\,\Phi_\mu(x) - a_1 a_4 x^2 =$$
$$= \mu a_1^{-1}\{[a_1 z + \Phi(x)]^2 + [a_1 a_2 x\,\Phi(x) - \Phi^2(x) - a_1^2 a_4 x^2]\} +$$
$$+ (1-\mu)\, a_1^{-1}\{[a_1 z + a_3 x]^2 + [a_1 a_2 a_3 - a_3^2 - a_1^2 a_4]\, x^2\} .$$

The second expression in brace brackets above is, by (1.11), non-negative, while the second expression in square brackets is precisely the function $\Psi(x)$ for which the estimate (3.6) holds. By setting $\Delta = a_1 a_2 a_3 - a_3^2 - a_1^2 a_4$ ($\Delta > 0$, in view of (1.11)) and by considering the interval $0 \leq \mu \leq \frac{1}{2}$, $\frac{1}{2} \leq \mu \leq 1$ separately it is easily verified that

$$W \geq \tfrac{1}{2} a_1^{-1} \min\{[a_1 z + \Phi(x)]^2 + \delta x^2 , [a_1 z + a_3 x]^2 + \Delta x^2\} - a_1^{-1} D_2 .$$

Thus, by an argument similar to that employed in § 3, we have that $\dot{V}_1$ satisfies

$$\dot{V}_1 \leq -D_{11}(x^2 + z^2) + D_{12}(1 + |x| + |y| + |u|)$$

for some constants D_{11} and D_{12}, and hence, $\dot{V}^* = \dot{V}_1 + \dot{V}_2^* + \dot{V}_3^*$, with $\lambda = 2D_{12}$ also has the property that

$$\dot{V}^* \leq -1 \qquad \text{if} \quad x^2 + y^2 + z^2 + u^2 \geq D_{16} ,$$

for some constant D_{16}. The desired boundedness property of solutions of (4.2) follows from this and (3.4) and, as remarked earlier, Theorem 3 now follows.

References

[1] A. I. Ogurcov (1959) – «Izv vyss Zaved Matematika», *10* (3), 200–209.
[2] J. O. Ezeilo and H. O. Tẹjumọla (1971) – «Ann. Mat. Pura Appl. (IV)», *88*, 207–216.
[3] H. O. Tẹjumọla (1968) – «Ann. Mat. Pura Appl. (IV)», *80*, 177–196.
[4] R. Reissig, G. Sansone and R. Conti (1963) – *Qualitative Theorie Nichtlinearer Differentialgleichungen*, Edizioni Cremonese, Roma.