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Harmonic summation of Jacobi series

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Analisi matematica (Serie Jacobi). — *Harmonic summation of Jacobi series.* Nota di D. P. GUPTA e R. S. CHOUDHARY, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori dimostrano un nuovo teorema sulla sommazione armonica delle serie di Jacobi.

INTRODUCTION

In a previous paper published in this Journal [1] one of the authors has proved a theorem on the Nörlund summability of the Jacobi series at the end point $x = 1$. It does not include the case of harmonic summability which is an important particular case of (N, p_n) summability. The object of the present paper is to give a separate treatment of this process.

1. Let $f(x)$ be a Lebesgue-measurable function such that the integral

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x) \, P_n^{(\alpha,\beta)}(x) \, dx \tag{1.1}$$

exists, $P_n^{(\alpha,\beta)}(x)$ being the n-th Jacobi polynomial. The Jacobi series associated with this function is

$$f(x) \sim \Sigma \, a_n \, P_n^{(\alpha,\beta)}(x) \tag{1.2}$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)}{2^{\alpha+\beta+1}} \, \frac{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\,\Gamma(n+\alpha+1)} \int_{-1}^{1} (1-y)^{\alpha} (1+y)^{\beta} f(y) \, P_n^{(\alpha,\beta)}(y) \, dy . \tag{1.3}$$

Let Σu_n be a given infinite series and $\{S_n\}$ be the sequence of the partial sums. The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k \, S_{n-k} \quad , \quad P_n = \sum_{i=0}^{i=n} p_i \neq 0 \, , \, p_0 > 0 \tag{1.4}$$

defines the (N, p_n)-mean of the sequence $\{S_n\}$. The series Σu_n or the sequence $\{S_n\}$ is said to be summable by Nörlund means to the sum S, if limit t_n exists and equals to S.

The important particular case of (N, p_n)-summability is the harmonic summability.

(*) Nella seduta dell'8 febbraio 1975.

When

(1.5) $$p_n = 1/(n+1)$$

a sequence $\{S_n\}$ is said to be summable by harmonic if

(1.6) $$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{S_{n-k}}{k+1}$$

exists.

Dealing with .... summability of Jacobi series at the end point of the interval $[-1, 1]$ Choudhary [1] has recently proved the following theorem.

Write

$$F(\varphi) = (f(\cos\varphi) - A)\left(\sin\frac{\varphi}{2}\right)^{2\alpha+1}\left(\cos\frac{\varphi}{2}\right)^{2\beta+1}$$

THEOREM. *Let* $\{p_n\}$ *be a real, non-negative, monotonic non-increasing sequence of coefficients* $\{p_n\}$ *such that* $P_n \to \infty$ *as* $n \to \infty$. *Then if*

(1.7) $$\int_0^t |F(\varphi)|\, d\varphi = o\left\{\frac{p(1/t)}{P(1/t)}\, t^{2\alpha+1}\right\}, \quad \text{as } t \to 0$$

and

(1.8) $$\Sigma_n \frac{n^{\alpha+1/2}}{P_n} < \infty$$

the series (1.2) *is summable* (N, p_n) *at the point* $x = 1$ *to the sum* A *provided* $-1/2 \leq \alpha < 1/2$, $\beta > -1/2$ *and the antipole condition*

$$\int_{-1}^{b} (1+x)^{\beta/2-3/4} |f(x)|\, dx < \infty$$

b fixed, is satisfied.

The above theorem does not cover the case of harmonic summability. The object of present paper is to investigate the problem of harmonic summability of Jacobi series at end point $x = 1$ of the interval.

The following theorem will be proved.

THEOREM. *If*

(1.11) $$F_1(t) \equiv \int_0^t |F(\varphi)|\, d\varphi = o\left(t^{\alpha+3/2}\right)$$

then the series (1.2) *is summable by harmonic means at the point* $x = +1$ *to the sum* A *provided* $-1 < \alpha < -1/2$, $\beta > -1/2$ *and the antipole condition*

(1.12) $$\int_{-1}^{b} (1+x)^{\beta/2-3/4} |f(x)|\, dx < \infty$$

b fixed, is satisfied.

Remark 1. A similar theorem may be proved for the other end point $x = -1$, the only modification being an interchange of the parameter α and β in the enunciation of the theorem.

Remark 2. The convergence problem for the range $-1 < \alpha < -1/2$, $\beta > -1/2$ was discussed by Kogbetliantz [3] and Obrechkoff [4]. Obrechkoff proved convergence almost everywhere for points in the Lebesgue set and therefore condition (1.11) seems to be a natural condition for harmonic summability.

2. We require the following lemmas for proving our theorem.

LEMMA 1 [[5], p. 167]. *For α, β arbitrary and real, c a fixed positive constant*:

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2}\, O(n^{-1/2}), & c/n \le \theta \le \pi/2, \\ O(n^{\alpha}), & 0 \le \theta \le c/n. \end{cases} \tag{2.1}$$

LEMMA 2 [[5], p. 190]. *For* $\alpha > -1$, $\beta > -1$, $c/n \le \theta \le \pi - y_n$,

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2}\, k(\theta) \left\{\cos(N\theta + \gamma) + \frac{O(1)}{n\sin\theta}\right\}, \tag{2.2}$$

where:

$$K(\theta) = \frac{1}{\sqrt{\pi}}\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}$$

$$N = n + \frac{\alpha+\beta+1}{2}, \qquad \gamma = -(\alpha + 1/2)\,\pi/2.$$

LEMMA 3. *For* $0 < t < \pi$ *and* $\alpha > -1$,

$$\left|\sum_{k=0}^{n-1} \frac{(n-k)^{\alpha+1/2}}{(k+1)} \cos(k+1)t\right| \le A(1 + \log 1/t)\, n^{\alpha+1/2}. \tag{2.3}$$

Proof. Hardy and Rogosinski [2] have shown that for $0 < t < \pi$,

$$\left|\Sigma \frac{\cos(k+1)t}{(k+1)}\right| < A(1 + \log 1/t).$$

We shall demonstrate the result for $-1 < \alpha < -1/2$, i.e. the range of α permissible in our theorem. However, as enunciated in the Lemma, the result is valid for all $\alpha > -1$. In fact, for $\alpha \ge -1/2$ the proof is easier and straightforwardly obtained by Abel's transformation.

In the case $-1 < \alpha < -1/2$, we write

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{(n-k)^{\alpha+1/2}}{k+1} \cos(k+1)t \\ &= \left[\sum_{k=0}^{[n/2]} + \sum_{k=[n/2]+1}^{n-1}\right] \frac{(n-k)^{\alpha+1/2}}{(k+1)} \cos(k+1)t \\ &= \Sigma_1 + \Sigma_2, \qquad \text{say.} \end{aligned} \tag{2.4}$$

By Abel's transformation

$$(2.5)\qquad |\Sigma_1| = \left| \sum_{k=0}^{[n/2]-1} \Delta\{(n-k)^{\alpha+1/2}\} \sum_{u=0}^{u=k} \frac{\cos(u+1)t}{u+1} + \right.$$

$$\left. + \left\{n - \left[\frac{n}{2}\right]\right\}^{\alpha+1/2} \sum_{u=0}^{[n/2]} \frac{\cos(u+1)t}{u+1} \right|$$

$$\leq A_1\, n^{\alpha+1/2}(1+\log 1/t) + A_2\, n^{\alpha+1/2}(1+\log 1/t)$$

$$= A_3\, n^{\alpha+1/2}(1+\log 1/t),$$

where A_i, $i = 1, 2, 3$ are constants.

Also

$$(2.6)\qquad |\Sigma_2| = \left| \sum_{k=[n/2]+1}^{n-1} \frac{(n-k)^{\alpha+1/2}}{(k+1)} \cos(k+1)t \right|$$

$$\leq A_4 \frac{1}{\left[\frac{n}{2}\right]+2} \sum_{k=[n/2]+1}^{n-1} (n-k)^{\alpha+1/2}$$

$$\leq \frac{A_5}{\left[\frac{n}{2}\right]+2} \left[(n-k)^{\alpha+3/2}\right]_{k=[n/2]+1}^{n-1}$$

$$= A_6\, n^{\alpha+1/2}, \qquad \text{since}\quad \alpha < -1/2.$$

(2.4), (2.5) and (2.6) lead to the required result.

LEMMA 4. *For all values of n and t and $\alpha > -1$,*

$$(2.7)\qquad \left| \sum_{k=0}^{n-1} \frac{(n-k)^{\alpha+1/2}}{(k+1)} \sin(k+1)t \right| \leq B\, n^{\alpha+1/2},$$

where B *is a constant.*

Proof. Titchmarsh [6] has shown that for all values of n and t:

$$\left| \sum_{k=0}^{n} \frac{\sin(k+1)t}{(k+1)} \right| \leq \pi/2 + 1\,.$$

The rest of the proof may be constructed on the lines of the proof of Lemma 3.

LEMMA 5. *For $0 \leq \varphi \leq c/n$, α, β arbitrary and real,*

$$(2.8)\qquad |N_n(\varphi)| \equiv \left| \frac{2^{\alpha+\beta+1}}{\log n} \sum_{k=0}^{n-1} \frac{\lambda_{n-k}}{k+1} P_{n-k}^{(\alpha+1,\beta)}(\cos\varphi) \right|$$

$$= O\left(n^{2\alpha+2}\right).$$

Proof. Since

$$\lambda_n \simeq \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)}\, n^{\alpha+1},$$

we have

$$| N_n(\varphi) | = \frac{O(1)}{\log n} \sum_{k=0}^{n-1} \frac{(n-k)^{\alpha+1}}{(k+1)} (n-k)^{\alpha+1}$$

$$= \frac{O(1)}{\log n} (n^{2\alpha+2}) \sum_{k=0}^{n-1} \frac{1}{k+1} \quad , \text{ as } \alpha > -1 ,$$

$$= O(n^{2\alpha+2}).$$

LEMMA 6. *If* $c/n \leq \varphi \leq \pi - y_n$, $-1 < \alpha < -1/2$, $\beta > -1$,

$$(2.9) \qquad | N_n(\varphi) | = O(1) \left[\left(\sin \frac{\varphi}{2}\right)^{-\alpha-3/2} \left(\cos \frac{\varphi}{2}\right)^{-\beta-1/2} n^{\alpha+1/2} \right] +$$

$$+ O(1) \left[\frac{1}{n \log n} \left(\sin \frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\cos \frac{\varphi}{2}\right)^{-\beta-3/2} \right].$$

Proof. Using the result of Lemma 2, we have

$$N_n(\varphi) = \frac{2^{\alpha+\beta+1}}{\log n} \sum_{k=0}^{n-1} \frac{\lambda_{n-k}}{(k+1)} \frac{(n-k)^{-1/2}}{\sqrt{\pi}} \left(\sin \frac{\varphi}{2}\right)^{-\alpha-3/2}$$

$$\left(\cos \frac{\varphi}{2}\right)^{-\beta-1/2} \left[\cos (N'\varphi + \gamma') + \frac{O(1)}{(n-k) \sin \varphi} \right],$$

where

$$N' = n + \frac{\alpha+\beta+2}{2} - k \qquad \text{and} \qquad \gamma' = -(\alpha + 3/2)\pi/2.$$

Let us write the asymptotic value of λ_{n-k}:

$$(2.10) \qquad N_n(\varphi) = \frac{1}{\sqrt{\pi} \log n \, \Gamma(\alpha+1)} \sum_{k=0}^{n-1} \frac{(n-k)^{\alpha+1/2}}{(k+1)} \left(\sin\left(\frac{\varphi}{2}\right)\right)^{-\alpha-3/2}$$

$$\left(\cos \frac{\varphi}{2}\right)^{-\beta-1/2} \left[\cos \left\{ \left((n-k) + \frac{2+\alpha+\beta}{2} \right) \varphi - \frac{\pi}{2} \left(\alpha + \frac{3}{2}\right) \right\} \right]$$

$$+ \frac{O(1)}{\sqrt{\pi} \log n \, \Gamma(\alpha+1)} \sum_{k=0}^{n-1} \frac{(n-k)^{\alpha-1/2}}{(k+1)} \left(\sin \frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\cos \frac{\varphi}{2}\right)^{-\beta-3/2}$$

$$= \Sigma_1 + \Sigma_2 , \qquad \text{say} .$$

Now:

$$\cos \left\{ \left(n - k + \frac{\alpha+\beta+2}{2} \right) \varphi - \frac{\pi}{2} \left(\alpha + \frac{3}{2}\right) \right\}$$

$$= \cos \left[\left\{ \left(n + \frac{\alpha+\beta+4}{2} \right) \varphi - \frac{\pi}{2} \left(\alpha + \frac{3}{2}\right) \right\} - \varphi (k+1) \right]$$

$$= \cos \{ (n'\varphi + \gamma') - \varphi (k+1) \}$$

$$= \cos (n'\varphi + \gamma') \cos (k+1) \varphi + \sin (n'\varphi + \gamma') \sin (k+1) \varphi ,$$

where

$$n' = n + \frac{\alpha + \beta + 4}{2} \qquad \text{and} \quad \gamma' = -\frac{\pi}{2}\left(\alpha + \frac{3}{2}\right).$$

Therefore, using the results of Lemma 3 and 4, we have

$$(2.11) \qquad |\Sigma_1| = \frac{\left(\sin\frac{\varphi}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\varphi}{2}\right)^{-\beta-1/2}}{\log n}\left[O\left(n^{\alpha+1/2}\right)\left(1 + \log\frac{1}{\varphi}\right) + O\left(n^{\alpha+1/2}\right)\right]$$

$$= \frac{\left(\sin\frac{\varphi}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\varphi}{2}\right)^{-\beta-1/2}}{\log n} O\left(n^{\alpha+1/2}\right)(1 + \log n).$$

Also

$$|\Sigma_2| = O(1)\left[\frac{\left(\sin\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\cos\frac{\varphi}{2}\right)^{-\beta-3/2}}{\log n}\sum_{k=0}^{n-1}\frac{(n-k)^{\alpha-1/2}}{k+1}\right],$$

but

$$\sum_{k=0}^{n-1}\frac{(n-k)^{\alpha-1/2}}{(k+1)} = \sum_{k=0}^{[n/2]}\frac{(n-k)^{\alpha-1/2}}{(k+1)} + \sum_{k=[n/2]+1}^{n-1}\frac{(n-k)^{\alpha-1/2}}{(k+1)}$$

$$\leq n^{\alpha-1/2}\sum_{k=0}^{[n/2]}\frac{1}{k+1} + \frac{1}{\left[\frac{n}{2}\right]+2}\sum_{k=[n/2]+1}^{n-1}(n-k)^{\alpha-1/2}$$

$$= O\left(n^{\alpha-1/2}\log n\right) + O(1/n) + O\left(n^{\alpha-1/2}\right)$$

$$= O(1/n), \qquad \text{since} \quad \alpha < -1/2.$$

Therefore

$$(2.12) \qquad |\Sigma_2| = O\left[\frac{\left(\sin\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\cos\frac{\varphi}{2}\right)^{-\beta-3/2}}{n\log n}\right].$$

Combining (2.11) and (2.12) we have the result of the Lemma.

LEMMA 7. *If* $\pi - c/n \leq \varphi \leq \pi$, $\alpha > -1$, $\beta > -1$,

$$(2.13) \qquad |N_n(\varphi)| = \left|\frac{2^{\alpha+\beta+1}}{\log n}\sum_{k=0}^{n-1}\frac{\lambda_{n-k}}{k+1}P_{n-k}^{(\alpha+1,\beta)}(\cos\varphi)\right|$$

$$= O\left(n^{\alpha+\beta+1}\right).$$

Proof.

$$|N_n(\varphi)| = \left|\frac{2^{\alpha+\beta+1}}{\log n}\sum_{k=0}^{n-1}\frac{\lambda_{n-k}}{k+1}(-1)^{n-k}P_{n-k}^{(\beta,\alpha+1)}(\cos t)\right|.$$

where $0 < t \leq c/n$,

$$= O\left(\frac{1}{\log n}\right) \sum_{k=0}^{n-1} \frac{(n-k)^{\alpha+\beta+1}}{k+1}$$

$$= O\left(\frac{1}{\log n}\right) \left[\sum_{k=0}^{[n/2]} \frac{(n-k)^{\alpha+\beta+1}}{k+1} + \sum_{[n/2]+1}^{n-1} \frac{(n-k)^{\alpha+\beta+1}}{k+1}\right]$$

$$= O\left(\frac{1}{\log n}\right) [n^{\alpha+\beta+1} \log n] + O\left(\frac{1}{\log n}\right) \frac{1}{n} [(n-k)^{\alpha+\beta+2}]_{[n/2]+1}^{n-1}$$

$$= O(n^{\alpha+\beta+1}).$$

LEMMA 8. (1) *The antipole condition* (1.12) *viz., the condition*

$$\int_{-1}^{b} (1+x)^{\beta/2-3/4} \, |f(x)| \, dx < \infty ,$$

implies for $\beta > -1/2$,

$$\int_{a=\cos^{-1} b}^{\pi} |f(\cos\theta) - A| \, (\cos\theta/2)^{\beta-1/2} \, d\theta < \infty , \tag{2.14}$$

which further implies

$$\int_{0}^{1/n} \theta^{\beta-1/2} \, |f(-\cos\theta) - A| \, d\theta = o(1) , \qquad n \to \infty .$$

Proof. The condition (1.12), on substituting $x = \cos\theta$, gives

$$\int_{a=\cos^{-1} b}^{\pi} |f(\cos\theta)| \, (\cos\theta/2)^{\beta-1/2} \, d\theta < \infty . \tag{2.16}$$

Since the integral

$$\int_{a}^{\pi} (\cos\theta/2)^{s} \, d\theta < \infty, \qquad \text{for } s > -1 ,$$

the integral

$$\int_{a}^{\pi} (\cos\theta/2)^{\beta-1/2} \, d\theta$$

exists under the conditions of the theorem.

Therefore, from (2.16) we obtain

$$\int_{a}^{\pi} (\cos\theta/2)^{\beta-1/2} \, |f(\cos\theta) - A| \, d\theta < \infty .$$

Substituting $\theta = \pi - t$, this gives

$$\int_0^{\pi-a} (\sin t/2)^{\beta-1/2} \, |f(-\cos t) - A| \, dt < \infty .$$

This means

$$\int_0^{\pi-a} t^{\beta-1/2} \, |f(-\cos t) - A| \, dt < \infty .$$

By the property of the Lebesgue integral, therefore:

$$\int_0^{1/n} t^{\beta-1/2} \, |f(-\cos t) - A| \, dt = o(1), \qquad \text{as} \quad n \to \infty .$$

3. Proof of the theorem

Following Obrechkoff [4], the n-th partial sum of the series (1.2) at the end point $x = 1$ is given by

$$S_n(1) = 2^{\alpha+\beta+1} \int_0^{\pi} \lambda_n \left(\sin \frac{\varphi}{2}\right)^{2\alpha+1} \left(\cos \frac{\varphi}{2}\right)^{2\beta+1} f(\cos \varphi) \, P_n^{(\alpha+1,\beta)}(\cos \varphi) \, d\varphi .$$

Consequently,

$$(3.1) \qquad S_n(1) - A = 2^{\alpha+\beta+1} \lambda_n \int_0^{\pi} F(\varphi) \, P_n^{(\alpha+1,\beta)}(\cos \varphi) \, d\varphi .$$

To prove the theorem it will be sufficient to show that (1)

$$(3.2) \qquad I(\varphi) \equiv \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{k+1} (S_{n-k} - A) = o(1) , \qquad n \to \infty .$$

Using (3.1) we get

$$I(\varphi) = \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{2^{\alpha+\beta+1}}{k+1} \int_0^{\pi} \lambda_{n-k} \, F(\varphi) \, P_{n-k}^{(\alpha+1,\beta)}(\cos \varphi) \, d\varphi$$

$$= \int_0^{\pi} F(\varphi) \, N_n(\varphi) \, d\varphi ,$$

where $N_n(\varphi)$ is as defined in (2.8).

We break the integral $I(\varphi)$ in four parts, as follows:

$$I(\varphi) = \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi} ,$$

(1) It is not necessary to consider the term $k = n$, since $S_0 - A = 0$.

δ being a suitably chosen, sufficiently small constant

$$= I_1 + I_2 + I_3 + I_4 , \qquad \text{say.} \tag{3.3}$$

We first consider I_1. From Lemma 5, we have

$$
\begin{aligned}
| I_1 | &= \int_0^{1/n} | F(\varphi) | \, | N_n(\varphi) | \, d\varphi \\
&= O(n^{2\alpha+2}) \int_0^{1/n} | F(\varphi) | \, d\varphi \\
&= O(n^{2\alpha+2}) \, o\left(\frac{1}{n^{\alpha+3/2}}\right), \qquad \text{from (1.11)} \\
&= o(n^{\alpha+1/2}) \\
&= o(1), \qquad \text{since } \alpha < -1/2 .
\end{aligned} \tag{3.4}
$$

From Lemma 6

$$
\begin{aligned}
| I_2 | &= \int_{1/n}^{\delta} | F(\varphi) | \, O\left\{\left(\sin \frac{\varphi}{2}\right)^{-\alpha-3/2} \left(\cos \frac{\varphi}{2}\right)^{-\beta-1/2} n^{\alpha+1/2}\right\} + \\
&\quad + \int_{1/n}^{\delta} | F(\varphi) | \, O\left\{\left(\sin \frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\cos \frac{\varphi}{2}\right)^{-\beta-3/2} \frac{1}{n \log n}\right\} \\
&= I_{2,1} + I_{2,2} , \qquad \text{say.}
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
| I_{2,1} | &= O(n^{\alpha+1/2}) \int_{1/n}^{\delta} | F(\varphi) | \, \varphi^{-\alpha-3/2} \, d\varphi \\
&= O(n^{\alpha+1/2}) \left[\varphi^{-\alpha-3/2} F_1(\varphi)\right]_{1/n}^{\delta} + \\
&\quad + O(n^{\alpha+1/2}) \int_{1/n}^{\delta} F_1(\varphi) \, \varphi^{-\alpha-5/2} \, d\varphi \\
&= O(n^{\alpha+1/2}) + O(n^{\alpha+1/2}) (n^{\alpha+3/2}) \, o\left[1/n^{\alpha+3/2}\right] + \\
&\quad + O(n^{\alpha+1/2}) \int_{1/n}^{\delta} o(\varphi^{\alpha+3/2}) \, \varphi^{-\alpha-5/2} \, d\varphi , \qquad \text{since } \delta
\end{aligned}
$$

is chosen sufficiently small.

$$= o(1), \qquad \text{since } \alpha < -1/2 . \tag{3.6}$$

Also

$$(3.7)\qquad |\mathrm{I}_{2,2}| = \mathrm{O}\left(\frac{1}{n\log n}\right)\int_{1/n}^{\delta} |\mathrm{F}(\varphi)|\,\varphi^{-\alpha-5/2}\,d\varphi$$

$$= \mathrm{O}\left(\frac{1}{n\log n}\right)[\mathrm{F}_1(\varphi)\,\varphi^{-\alpha-5/2}]_{1/n}^{\delta} + \mathrm{O}\left(\frac{1}{n\log n}\right)\int_{1/n}^{\delta} \mathrm{F}_1(\varphi)\,\varphi^{-\alpha-7/2}\,d\varphi$$

$$= \mathrm{O}\left(\frac{1}{n\log n}\right) + \mathrm{O}\left(\frac{1}{n\log n}\right) n^{\alpha+5/2}\left\{o\left(\frac{1}{n^{\alpha+3/2}}\right)\right\} +$$

$$+ \mathrm{O}\left(\frac{1}{n\log n}\right)\int_{1/n}^{\delta} \{o(\varphi^{\alpha+3/2})\}\,\varphi^{-\alpha-7/2}\,d\varphi$$

$$= o(1) + o\left(\frac{1}{n\log n}\right)\int_{1/n}^{\delta} \frac{1}{\varphi^2}\,d\varphi$$

$$= o(1)\,.$$

Coming now to I_3, we have

$$|\mathrm{I}_3| = \left|\int_{\delta}^{\pi-1/n} \mathrm{F}(\varphi)\,\mathrm{N}_n(\varphi)\,d\varphi\right|$$

$$= \mathrm{O}(n^{\alpha+1/2})\int_{\delta}^{\pi-1/n} |\mathrm{F}(\varphi)|\left(\sin\frac{\varphi}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\varphi}{2}\right)^{-\beta-1/2} d\varphi +$$

$$+ \mathrm{O}\left(\frac{1}{n\log n}\right)\int_{\delta}^{\pi-1/n} |\mathrm{F}(\varphi)|\left(\sin\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\cos\frac{\varphi}{2}\right)^{-\beta-3/2} d\varphi$$

$$= \mathrm{O}(n^{\alpha+1/2})\int_{\delta}^{\pi-1/n} |\mathrm{F}(\varphi)|\left(\cos\frac{\varphi}{2}\right)^{\beta+1/2}\left(\cos\frac{\varphi}{2}\right)^{-2\beta-1} d\varphi +$$

$$+ \mathrm{O}\left(\frac{1}{n\log n}\right)\int_{\delta}^{\pi-1/n} |\mathrm{F}(\varphi)|\left(\cos\frac{\varphi}{2}\right)^{\beta-1/2}\left(\cos\frac{\varphi}{2}\right)^{-2\beta-1} d\varphi\,.$$

but

$$\mathrm{F}(\varphi) = [f(\cos\varphi) - \mathrm{A}]\left(\sin\frac{\varphi}{2}\right)^{2\alpha+1}\left(\cos\frac{\varphi}{2}\right)^{2\beta+1},$$

$$(3.8)\qquad |\mathrm{I}_3| = \mathrm{O}(n^{\alpha+1/2})\int_{\delta}^{\pi-1/n} |f(\cos\varphi) - \mathrm{A}|\left(\cos\frac{\varphi}{2}\right)^{\beta-1/2}\left(\cos\frac{\varphi}{2}\right) d\varphi +$$

$$+ \mathrm{O}\left(\frac{1}{n\log n}\right)\int_{\delta}^{\pi-1/n} |f(\cos\varphi) - \mathrm{A}|\left(\cos\frac{\varphi}{2}\right)^{\beta-1/2} d\varphi$$

$$= O(n^{\alpha+1/2}) + O(1/n \log n), \qquad \text{from} \quad (2.14)$$

$$= o(1), \qquad \text{since} \quad \alpha < -1/2 .$$

Finally

$$(3.9) \qquad |I_4| = \left| \int_{\pi-1/n}^{\pi} F(\varphi) N_n(\varphi) d\varphi \right|$$

$$= \left| \int_0^{1/n} F(\pi - t) N_n(\pi - t) dt \right|$$

$$= O(n^{\alpha+\beta+1}) \int_0^{1/n} |f(-\cos t) - A| \left(\cos \frac{t}{2}\right)^{2\alpha+1} \left(\sin \frac{t}{2}\right)^{2\beta+1} dt$$

$$= O(n^{\alpha+\beta+1}) \int_0^{1/n} |f(-\cos t) - A| \, t^{2\beta+1} dt$$

$$= O(n^{\alpha-1/2}) \int_0^{1/n} |f(-\cos t) - A| \, t^{\beta-1/2} dt$$

$$= o(n^{\alpha-1/2}), \qquad \text{from} \quad (2.15)$$

$$= o(1), \qquad \text{since} \quad \alpha < -1/2 .$$

Combining (3.3), (3.4) $\cdots$ (3.9), we have

$$I(\varphi) = o(1) .$$

This completes the proof of the theorem.

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