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Dual cross-sectional measures

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Geometria. — *Dual cross-sectional measures.* Nota di ERWIN LUTWAK, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si definiscono « quermass integrali » duali a mezzo di formole di Kubota duali. Si esaminano le relazioni fra questi integrali ed i funzionali di Minkowski; in particolare, si trovano le disuguaglianze che corrispondono per dualità a quelle classiche.

The integral formulas of Kubota [3] permit a simple recursive definition of the $n+1$ cross-sectional measures $W_0, W_1, \cdots, W_n$ in Euclidean n-space, R^n. In a one dimensional space, for convex compact A, $W_0(A)$ is defined to be the length of A while $W_1(A)$ is defined to be 2. After defining the cross-sectional measures in $(n-1)$-space they are defined in Euclidean n-space by letting $W_0(A)$ equal $V(A)$, the n-dimensional volume of A, and letting

$$W_i(A) = \frac{1}{n\omega_{n-1}} \int_{\Omega} W'_{i-1}(A \mid P_u)\, dS(u) \qquad [i > 0]. \tag{1}$$

In this integral Ω is the surface of the unit ball U in R^n while dS denotes the area element on Ω. $A \mid P_u$ is the projection of A onto the hyperplane P_u which is perpendicular to $u \in \Omega$ and passes through the origin. W'_{i-1} denotes the $(i-1)$-th cross-sectional measure in $(n-1)$-space while ω_{n-1} denotes the volume of the unit ball in R^{n-1}.

The dual cross-sectional measures of the title refer to the measures obtained when the definition of the cross-sectional measures is altered by replacing $A \mid P_u$ in (1) by its dual $A \cap P_u$. These measures arise naturally in the examination of a radial addition that is analogous to Minkowski addi-

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tion. They appear as coefficients in a polynomial expression analogous to the Steiner polynomial expression for the volume of a parallel body [3]. In addition, the natural extension of these measures can be used to examine a harmonic addition previously considered by Firey [1, 2], Steinhardt [7, p. 15] and Rockafellar [6, p. 21].

The setting for this paper will be R^n. Compact convex sets with non-empty interiors will be called convex bodies. The space of all convex bodies which contain the origin in their interior, endowed with the Hausdorff topology, will be denoted by K_n. For a convex body A, we use H_A and F_A to denote its support and distance function, respectively. A compact set A is called a star body with respect to a point a if for every c in A the line $\{ta + (1 - t)c \mid 0 < t \leq 1\}$ lies in the interior of A. The set of all star bodies with respect to the origin will be denoted by S_n. Associated with a star body $A \in S_n$ is a radial function ρ_A defined on Ω by:

$$\rho_A(u) = \mathrm{Sup}\{\lambda > 0 \mid \lambda u \in A\} \qquad [u \in \Omega] .$$

The radial function of a star body in S_n is a positive, continuous real-valued function on Ω. Conversely, a positive, continuous real-valued function on Ω is the radial function of a unique star body in S_n. A metric d can be defined in S_n by letting

$$d(A, B) = \mathrm{Sup}_{u \in \Omega} \mid \rho_A(u) - \rho_B(u) \mid \qquad [A, B \in S_n] .$$

It is easy to verify that S_n endowed with the topology induced by this metric has K_n as a subspace.

While the cross-sectional measures are defined for compact convex sets, the dual cross-sectional measures $\tilde{W}_0, \tilde{W}_1, \cdots, \tilde{W}_n$ will be defined for star bodies in S_n.

DEFINITION 1. In R^1 the dual cross-sectional measures are defined by

$$\tilde{W}_0(A) = V(A) \qquad \tilde{W}_1(A) = \omega_1 \qquad [A \in S_1] .$$

After defining the dual cross-sectional measures in Euclidean $(n - 1)$-space they are defined in R^n by letting $\tilde{W}_0(A) = V(A)$ and

$$(2) \qquad \tilde{W}_i(A) = \frac{1}{n\omega_{n-1}} \int_\Omega \tilde{W}'_{i-1}(A \cap P_u)\, dS(u) \qquad [i > 0 \quad A \in S_n]$$

where $\tilde{W}'_{i-1}$ denotes the $(i - 1)$-th dual cross-sectional measure in Euclidean $(n - 1)$-space.

Comparing (2) with (1), we obtain:

THEOREM 1.

$$\tilde{W}_i(A) \leq W_i(A) \qquad [0 < i < n \quad A \in K_n]$$

with equality if and only if A *is an n-ball (centered at the origin).*

The i-th dual cross-sectional measure $\tilde{W}_i$ is a map

$$\tilde{W}_i : \; S_n \to R .$$

It is continuous, bounded, additive, positive, rotation invariant, homogeneous of degree $n - i$ and monotone under set inclusion. All of these properties are simple consequences of our next theorem which describes the dual cross-sectional measures of a star body as means of powers of its radial function.

THEOREM 2.

$$\tilde{W}_i(A) = \frac{1}{n} \int_{\Omega} \rho_A^{n-i}(u) \, dS(u) \qquad [A \in S_n] .$$

The proof follows by induction on the dimension of the space using standard techniques (see Hadwiger [3, p. 212]).

The cross-sectional measures satisfy the cyclic inequality [3, p. 282]:

$$W_j^{k-i}(A) \geq W_i^{k-j}(A) \, W_k^{j-i}(A) \qquad [i < j < k \quad A \in K_n].$$

As a simple consequence of Hölder's Inequality [4, p. 140] we have:

THEOREM 3.

$$\tilde{W}_j^{k-i}(A) \leq \tilde{W}_i^{k-j}(A) \, \tilde{W}_k^{j-i}(A) \qquad [i < j < k \quad A \in S_n]$$

with equality if and only if A *is an n-ball (centered at the origin).*

We note, that for convex bodies in K_n, Theorem 3 is a consequence of a general inequality between dual mixed volumes that was obtained by us in [5].

The Minkowski sum $A + B$ of two convex bodies A and B can be defined by the equation

$$H_{A+B} = H_A + H_B .$$

Given two star bodies $A, B \in S_n$ we define the radial sum $A \otimes B$ by:

DEFINITION 2.

$$\rho_{A \otimes B} = \rho_A + \rho_B \qquad [A, B \in S_n] .$$

The Brunn-Minkowski Theorem [3, p. 187] states that:

$$V^{1/n}(A + B) \geq V^{1/n}(A) + V^{1/n}(B) \qquad [A, B \in K_n]$$

with equality if and only if A and B are homothetic. A simple application of the Minkowski Inequality [4, p. 146] yields:

THEOREM 4.

$$V^{1/n}(A \otimes B) \leq V^{1/n}(A) + V^{1/n}(B) \qquad [A, B \in S_n]$$

with equality if and only if A *is a dilation of* B *(with the origin as the center of dilation).*

For a convex body A and a scalar $\mu > 0$ the parallel body A_μ is defined to be $A + \mu U$. For a star body $A \in S_n$ and $\mu > 0$ we define the radial body ${}_\mu A$ by:

DEFINITION 3.

$$ {}_\mu A = A \otimes \mu U \qquad [\mu > 0 \quad A \in S_n] . $$

For the volume of the parallel body A_μ we have the Steiner polynomial expression [3, p. 214]:

$$ V(A_\mu) = \sum_{i=0}^{n} \binom{n}{i} W_i(A)\, \mu^i \qquad [\mu > 0 \quad A \in K_n] . $$

Just as the cross-sectional measures appear as coefficients in the polynomial expression of $V(A_\mu)$, the dual cross-sectional measures appear as coefficients in a polynomial expression of $V({}_\mu A)$.

THEOREM 5.

$$ V({}_\mu A) = \sum_{i=0}^{n} \binom{n}{i} \tilde{W}_i(A)\, \mu^i \qquad [\mu > 0 \quad A \in S_n] . $$

To prove this we merely note that $\rho_{\mu A} = \rho_A + \mu$.

Combining the definition of the surface area of a convex body [3, p. 184] with the Cauchy area formula [3, p. 208] we obtain:

$$ \lim_{\mu \to 0} [V(A_\mu) - V(A)]/\mu = \frac{1}{\omega_{n-1}} \int_\Omega V'(A \mid P_u)\, dS(u) \qquad [A \in K_n] $$

where V' denotes the volume in Euclidean $(n-1)$-space. As a direct consequence of Theorem 5 we have:

COROLLARY.

$$ \lim_{\mu \to 0} [V({}_\mu A) - V(A)]/\mu = \frac{1}{\omega_{n-1}} \int_\Omega V'(A \cap P_n)\, dS(u) \qquad [A \in S_n] . $$

As presented in Definition 1 the dual cross-sectional measures $\tilde{W}_i$ have indices i restricted to integer values between zero and n. However, Theorem 1 points to a natural extension of the definition so that the $\tilde{W}_i$ are defined for all real indices.

DEFINITION 1*.

$$ \tilde{W}_i(A) = \int_\Omega \rho_A^{n-i}(u)\, dS(u) \qquad [i \in R \quad A \in S_n] . $$

The new $\tilde{W}_i$ are also positive, continuous, additive, rotation invariant and homogeneous of degree $n - i$. However, they are bounded and monotone only for $i \leq n$. Theorem 3 remains unaltered if we allow the indices of the dual cross-sectional measures to range over all real numbers.

With extended indices the dual cross-sectional measures can be used to examine a harmonic addition considered by Firey [1, 2], Steinhardt [7, p. 15] and Rockafellar [6, p. 21].

The harmonic sum $A \times B$ of two convex bodies $A, B \in K_n$ is defined by:

DEFINITION 4.

$$F_{A\times B} = F_A + F_B \qquad [A, B \in K_n].$$

We note that, while we chose not to do so, harmonic addition could have been defined in S_n by letting $\rho_{A\times B} = (\rho_A^{-1} + \rho_B^{-1})^{-1}$. Both definitions coincide in K_n.

The following dual to the Brunn-Minkowski theorem is due to Firey [1] and Steinhardt [7]:

THEOREM 6.

$$V^{-1/n}(A\times B) \geq V^{-1/n}(A) + V^{-1/n}(B) \qquad [A, B \in K_n]$$

with equality if and only if A *is a dilation of* B (*with the origin as the center of dilation*).

The scalar product μA of a convex body A and a scalar $\mu > 0$ can be defined by the equation $H_{\mu A} = \mu H_A$. Analagously, we define a harmonic scalar product $\mu \circ A$ by:

DEFINITION 5.

$$F_{\mu\circ A} = \mu\, F_A \qquad [\mu > 0 \quad A \in K_n].$$

We note that harmonic scalar products could have been defined for star bodies $A \in S_n$ by letting $\rho_{\mu\circ A} = \mu^{-1}\rho_A$. Both definitions coincide in K_n.

Analogous to the definition of the parallel body we define the harmonic body A^μ by:

DEFINITION 6.

$$A^\mu = A \times \mu \circ U \qquad [\mu > 0 \quad A \in K_n].$$

Our last theorem shows that, for small μ, the extended dual cross-sectional measures appear as coefficients in an expression for $V(A^\mu)$.

THEOREM 7.

$$V(A^\mu) = \sum_{i=0}^{\infty} \binom{-n}{i} \tilde{W}_{-i}(A)\, \mu^i \qquad [\mu < \operatorname{Inf} \rho_A].$$

The proof is a simple exercise involving the use of the binomial theorem.

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