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On wave solutions of the field equations of general relativity

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Relatività. — *On wave solutions of the field equations of general relativity.* Nota di KRISHNA BIHARI LAL e MUSTAKEEM, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Ricerca delle soluzioni d'onda per la metrica relativistica (1.1).

1. INTRODUCTION

Recently Lal and Mustakeem [1] (1) have found out the wave solutions of Kilmister and Newmann's weakened field equations in the space time

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 + 2\,A\,dz\,dt + 2\,B\,dx\,dy\,, \tag{1.1}$$

where $A = A(z, t)$ and $B = B(x, y)$. In this paper using the metric (1.1) an attempt has been made to find the wave solutions of the field equations of general relativity

$$R_{ij} = -8\pi E_{ij}\,, \qquad (i, j = 1, 2, 3, 4)\,, \tag{1.2}$$

where R_{ij} is the Ricci tensor of the space-time (1.1) and E_{ij} is the electromagnetic tensor defined by

$$E_{ij} = \frac{1}{4} g_{ij} F_{kl} F^{kl} - F_{ik} F_{jl} g^{kl}, \tag{1.3}$$

where F_{ij} is the antisymmetric electromagnetic field tensor satisfying the generalized Maxwell's equations

$$(i)\quad F_{ij;k} + F_{jk;i} + F_{ki;j} = 0\,, \qquad (ii)\quad F^{ij}_{;j} = 0\,. \tag{1.4}$$

Here a semi-colon and a comma followed by an index denote covariant and usual partial derivatives respectively.

2. CALCULATION OF CHRISTOFFEL SYMBOLS AND THE RICCI TENSOR

The non-vanishing components of the contravariant tensor g^{ij} as calculated from the line element (1.1) are

$$\begin{aligned} g^{11} = g^{22} = g^{12}/B &= -1/(1 - B^2)\,, \\ -g^{33} = g^{44} = g^{34}/A &= 1/(1 + A^2) \end{aligned} \tag{2.1}$$

(*) Nella seduta dell'11 gennaio 1975.

(1) Numbers in brackets refer to the references at the end of the paper.

and the surviving components of the Christoffel symbols of the second kind $\left\{ {k \atop ij} \right\}$ are given by

(2.2)
$$\begin{aligned}
\left\{ {1 \atop 11} \right\} &= -BB_1/(1-B^2) \quad , & \left\{ {3 \atop 33} \right\} &= AA_3/(1+A^2) \,, \\
\left\{ {2 \atop 11} \right\} &= -B_1/(1-B^2) \quad , & \left\{ {4 \atop 33} \right\} &= A_3/(1+A^2) \,, \\
\left\{ {1 \atop 22} \right\} &= -B_2/(1-B^2) \quad , & \left\{ {3 \atop 44} \right\} &= -A_4/(1+A^2) \,, \\
\left\{ {2 \atop 22} \right\} &= -BB_2/(1-B^2) \quad , & \left\{ {4 \atop 44} \right\} &= AA_4/(1+A^2) \,;
\end{aligned}$$

where B_1 , B_2 , A_3 , A_4 stand for $\frac{\partial B}{\partial x} , \frac{\partial B}{\partial y} , \frac{\partial A}{\partial z} , \frac{\partial A}{\partial t}$ respectively. The non-vanishing components of the curvature tensor

$$R_{ijlm} \, (= -R_{jilm} = -R_{ijml} = R_{lmij})$$

are given by

(2.3)
$$\begin{aligned}
R_{1212} &= B_{12} + BB_1 B_2/(1-B^2) \,, \\
R_{3434} &= A_{34} - AA_3 A_4/(1+A^2) \,,
\end{aligned}$$

while the non-vanishing components of the Ricci tensor R_{ij} as obtained from (2.3) by contraction with the help of (2.1) are given by

(2.4)
$$\begin{aligned}
R_{11} = R_{22} = -R_{12}/B &= B_{12}/(1-B^2) + BB_1B_2/(1-B^2)^2, \\
-R_{33} = R_{44} = R_{34}/A &= A_{34}/(1+A^2) - AA_3A_4/(1+A^2)^2.
\end{aligned}$$

3. The electromagnetic field

Maxwell's equation (1.4) (*i*) is identically satisfied if

(3.1)
$$F_{ij} = \partial x^j \, k_i - \partial x^i \, k_j \qquad \left(\partial x^j = \frac{\partial}{\partial x^j} \, , \text{etc.} \right)$$

where

$$k_i = (-F \, , -G \, , -H \, , \Phi) \, .$$

Since the electric and magnetic forces with components X , Y , Z and α , β , γ respectively, together form the curl of the electromagnetic potentials F , G , H , Φ; the complete scheme [2] for F_{ij} is given by

(3.2)
$$F_{ij} = \begin{bmatrix} 0 & -\gamma & \beta & -X \\ \gamma & 0 & -\alpha & -Y \\ -\beta & \alpha & 0 & -Z \\ X & Y & Z & 0 \end{bmatrix}$$

Now we assume that F_{ij} is transverse electromagnetic, and that its components are functions of $x, y, (z-t)$. Consequently there shall be no surviving components of both electric and magnetic fields in the direction of wave propagation, i.e. along z-axis, and we shall have

$$(3.3) \qquad a)\ F_{34} = 0 \qquad \text{and} \qquad b)\ F_{12} = 0$$

which gives

$$(3.4) \qquad a)\ \overline{H} = \overline{\Phi} \qquad \text{and} \qquad b)\ \partial_y F = \partial_x G$$

respectively where a bar over H and Φ denotes differentiation with respect to $(z-t)$.

Using (3.4) in (3.1), we get

$$(3.5) \qquad a)\ F_{13} + F_{14} = c_1 \qquad \text{and} \qquad b)\ F_{23} + F_{24} = c_2$$

where c_1 and c_2 are constants. Assuming $c_1 = c_2 = 0$, (3.5) reduces to

$$(3.6) \qquad a)\ -F_{13} = F_{14} = \sigma \qquad \text{and} \qquad b)\ F_{23} = -F_{24} = \rho$$

where σ and ρ are obviously the functions of $x, y, (z-t)$. Using (3.3) and (3.6) in (3.2) we find

$$(3.7) \qquad F_{ij} = \begin{bmatrix} 0 & 0 & -\sigma & \sigma \\ 0 & 0 & \rho & -\rho \\ \sigma & -\rho & 0 & 0 \\ -\sigma & \rho & 0 & 0 \end{bmatrix}.$$

Again, with the help of (3.1) and (3.4), it can easily be seen that

$$\partial_y F_{13} - \partial_x F_{23} = 0$$

which by (3.6) gives

$$(3.8) \qquad \partial_x \rho + \partial_y \sigma = 0 .$$

The contravariant components of the electromagnetic field tensor F^{ij} from (3.7) and (2.1) are, therefore, found to be

$$(3.9) \quad F^{ij} = \begin{bmatrix} 0 & 0 & \frac{(1+A)(B\rho-\sigma)}{m} & \frac{(1-A)(B\rho-\sigma)}{m} \\ 0 & 0 & \frac{-(1+A)(B\sigma-\rho)}{m} & \frac{-(1-A)(B\sigma-\rho)}{m} \\ \frac{-(1+A)(B\rho-\sigma)}{m} & \frac{(1+A)(B\sigma-\rho)}{m} & 0 & 0 \\ \frac{-(1-A)(B\rho-\sigma)}{m} & \frac{(1-A)(B\sigma-\rho)}{m} & 0 & 0 \end{bmatrix}$$

where

$$m = (1 + A^2)(1 - B^2) .$$

The components of the dual tensor F^*_{ij} of the electromagnetic field tensor F_i are given by $F^*_{ij} = \frac{1}{2} E_{ijkl} F^{kl}$, where E_{ijkl} is antisymmetric with respect to each pair of indices and $E_{1234} = \sqrt{-g} = \sqrt{m}$, $(g = \det(g_{ij}))$.

The components of F^*_{ij} and the corresponding contravariant tensor F^{*ij} are given by

$$F^*_{ij} = \begin{bmatrix} 0 & 0 & \frac{(1-A)(B\sigma-\rho)}{\sqrt{m}} & \frac{-(1+A)(B\sigma-\rho)}{\sqrt{m}} \\ 0 & 0 & \frac{(1-A)(B\rho-\sigma)}{\sqrt{m}} & \frac{-(1+A)(B\rho-\sigma)}{\sqrt{m}} \\ \frac{-(1-A)(B\sigma-\rho)}{\sqrt{m}} & \frac{-(1-A)(B\rho-\sigma)}{\sqrt{m}} & 0 & 0 \\ \frac{(1+A)(B\sigma-\rho)}{\sqrt{m}} & \frac{(1+A)(B\rho-\sigma)}{\sqrt{m}} & 0 & 0 \end{bmatrix}$$

(3.10)

$$F^{*ij} = \begin{bmatrix} 0 & 0 & -\rho/\sqrt{m} & -\rho/\sqrt{m} \\ 0 & 0 & -\sigma/\sqrt{m} & -\sigma/\sqrt{m} \\ \rho/\sqrt{m} & \sigma/\sqrt{m} & 0 & 0 \\ \rho/\sqrt{m} & \sigma/\sqrt{m} & 0 & 0 \end{bmatrix} .$$

From (3.7), (3.9) and (3.10) it is easily seen that $F_{ij} F^{ij} \neq 0$, $F^*_{ij} F^{ij} = 0$, as such the electromagnetic field is not null in the sense of Synge [3]. Next on substituting the values from (2.1), (3.7) and (3.9) in to (1.3), the non-vanishing components of electromagnetic energy tensor are given by

$$(3.11) \quad \begin{cases} E_{11} = L + \dfrac{2A\sigma^2}{(1+A^2)}, \\ E_{12} = -BL - \dfrac{2\rho\sigma A}{(1+A^2)}, \\ E_{22} = L + \dfrac{2A\rho^2}{(1+A^2)}, \\ E_{33} = L + \dfrac{(\rho^2 - 2B\rho\sigma + \sigma^2)}{(1-B^2)}, \\ E_{34} = -AL - \dfrac{(\rho^2 - 2B\rho\sigma + \sigma^2)}{(1-B^2)}, \\ E_{44} = -L + \dfrac{(\rho^2 - 2B\rho\sigma + \sigma^2)}{(1-B^2)}; \end{cases}$$

where

$$L = \frac{(B\rho - \sigma)\sigma A}{(1+A^2)(1-B^2)} + \frac{(B\sigma - \rho)\rho A}{(1+A^2)(1-B^2)} .$$

4. SOLUTIONS OF THE FIELD EQUATIONS

Substituting the values of R_{ij} and E_{ij} from (2.4) and (3.11) into the field equation (1.2), we have

$$\frac{B_{12}}{(1-B^2)}+\frac{BB_1B_2}{(1-B^2)^2}=-8\pi\left(L+\frac{2A\sigma^2}{(1+A^2)}\right), \tag{4.1 a}$$

$$\frac{B_{12}}{(1-B^2)}+\frac{BB_1B_2}{(1-B^2)^2}=-8\pi\left(L+\frac{2A\rho^2}{(1+A^2)}\right), \tag{4.1 b}$$

$$\frac{A_{34}}{(1+A^2)}-\frac{AA_3A_4}{(1+A^2)^2}=8\pi\left(L+\frac{(\rho^2-2B\rho\sigma+\sigma^2)}{(1-B^2)}\right), \tag{4.1 c}$$

$$\frac{A_{34}}{(1+A^2)}-\frac{AA_3A_4}{(1+A^2)^2}=-8\pi\left\{-L+\frac{(\rho^2-2B\rho\sigma+\sigma^2)}{(1-B^2)}\right\}, \tag{4.1 d}$$

$$\frac{BB_{12}}{(1-B^2)}+\frac{B^2B_1B_2}{(1-B^2)^2}=-8\pi\left(BL+\frac{2A\rho\sigma}{(1+A^2)}\right), \tag{4.1 e}$$

$$\frac{AA_{34}}{(1+A^2)}-\frac{A^2A_3A_4}{(1+A^2)}=8\pi\left\{AL+\frac{(\rho^2-2B\rho\sigma+\sigma^2)}{(1-B^2)}\right\}. \tag{4.1 f}$$

From (4.1 *a*) and (4.1 *b*) we have $\rho=\pm\sigma$.

Case (*i*). When $\rho=\sigma$, equations (4.1 *c*)–(4.1 *f*) reduce to

$$\begin{cases}\dfrac{B_{12}}{(1-B^2)}+\dfrac{BB_1B_2}{(1-B^2)^2}=-\dfrac{16\pi\sigma^2A}{(1+A^2)(1+B)B}, \\[2ex] \dfrac{A_{34}}{(1+A^2)}-\dfrac{AA_3A_4}{(1+A^2)^2}=-\dfrac{16\pi\sigma^2A}{(1+A^2)(1+B)}.\end{cases} \tag{4.2}$$

The field equation (1.4) (*i*) is satisfied identically by the components of F_{ij} while the field equation (1.4) (*ii*) is satisfied if

$$\begin{cases}(B\rho_1+B_1\sigma-\sigma_1)-\sigma BB_1/(1+B)=0, \\ (B\sigma_2+B_2\sigma-\rho_2)-\sigma BB_2/(1+B)=0.\end{cases} \tag{4.3}$$

Thus the wave solutions of the field equations (1.2) and (1.4) are composed of g_{ij} given by (1.1) and F_{ij} given by (3.7) with (3.8) under the conditions (4.2) and (4.3).

Case (*ii*). When $\rho=-\sigma$, equations (4.1 *c*)–(4.1 *f*) reduce to

$$\begin{cases}\dfrac{B_{12}}{(1-B^2)}+\dfrac{BB_1B_2}{(1-B^2)^2}=-\dfrac{16\pi\sigma^2A(1-2B)}{(1+A^2)(1-B)B}, \\[2ex] \dfrac{A_{34}}{(1+A^2)}-\dfrac{AA_3A_4}{(1+A^2)^2}=-\dfrac{16\pi\sigma^2A}{(1+A^2)(1-B)}.\end{cases} \tag{4.4}$$

The field equation (1.4) (*i*) is satisfied identically by the components of F_{ij} while the field equation (1.4) (*ii*) is satisfied if

$$\begin{cases}-(1+A)\sigma_3-\sigma A_3+(1+A)\sigma AA_3/(1+A^2)=0 \\ -(1-A)\sigma_4+\sigma A_4+(1-A)\sigma AA_4/(1+A^2)=0\end{cases} \tag{4.5}$$

and

$$(4.6) \qquad \begin{cases} (B\rho_1 - B_1\sigma - \sigma_1) - \sigma BB_1/(1 - B) = 0 \\ (B\sigma_2 + B_2\sigma - \rho_2) + \sigma BB_2/(1 - B) = 0 . \end{cases}$$

Thus, in this case the wave solutions of the field equations (1.2) and (1.4) are composed of g_{ij} given by (1.1) and F_{ij} given by (3.7) with (3.8) under the conditions (4.4), (4.5) and (4.6).

REFERENCES

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