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Characteristic classes of S^1-pseudo-actions in fiber bundles. Nota II. Chern classes

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Topologia algebrica. — *Characteristic classes of* S^1*–pseudo-actions in fiber bundles*. Nota II. *Chern classes* (*). Nota di NICOLAE TELEMAN, presentata (**) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Nella presente Nota continuiamo lo studio delle classi caratteristiche $\mathscr{C}_i$ definite da noi in [3]. Si dimostra che le classi $\mathscr{C}_i$ generalizzano le classi di Chern. Sottolineiamo il fatto che le classi $\mathscr{C}_i$ soddisfano la dualità di Whitney modulo classi di torsione di ordine dispari. Le classi $\mathscr{C}_i$ dei fibrati sferici associati ai fibrati vettoriali complessi coincidono con le classi di Chern. Osserviamo inoltre che le classi $\mathscr{C}_i$ non sono necessariamente di dimensione pari.

§ 1. INTRODUCTION

In our Note I [3] we have defined the classes $\mathscr{C}_i$ on the category of fibre bundles with S^1-pseudo-action.

In this Note we prove that the classes $\mathscr{C}_i$ generalize the Chern classes, although $\mathscr{C}_i$ not always are dimensional classes.

The intent of this Note is to prove the following Theorems A and B.

We begin with the following:

1.1. DEFINITION. *Let* $\xi = (E, \pi, B, F, S)$ *be a fibre bundle with* S^1*-pseudo-action* S, *and let* $f: B_1 \to B$ *be a continuous map; the "induced fibre bundle with* S^1*-pseudo-action",* $f^* \xi$, *is* $f^* \xi = (f^* E, f^* \pi, B_1, F, f^* S)$, *where* $(f^* E, f^* \pi, B_1)$ *is the induced fibre bundle and* $f^* S$ *is defined as follows: if* x *is an arbitrary point in* B_1, $(f^* \pi)^{-1} \{x\} = \pi^{-1} (f(x))$ *and we supply* $(f^* \pi)^{-1} \{x\}$ *with the* S^1*-pseudo-action which is defined in* $\pi^{-1} (f(x))$.

Let R be a commutative ring.

1.2. THEOREM A. (o). *If* $\xi = (E, \pi, B, F, H) \in \mathscr{B}^n_R (B, F, Z_2, S^1)$, *(see* § 2 [3]) *then:*

$$(1)\quad \begin{aligned} &\mathscr{C}_{n-2i+2} (\xi) = s^* \omega_{2i} (\Sigma^2 \xi, k_p^{2r})^{(*)} \\ &\mathscr{C}_{n-2i+2} (\xi) \in \mathscr{H}^{n-2i+2} (B, \mathscr{H}_{n-1} (E, R)), \quad \mathscr{C}_k (\xi) = 0 \textit{ for } k \neq n - 2i + 2,\ i \in \mathbf{Z}. \end{aligned}$$

(i) *If* ξ *is fibre bundle with* S^1*-pseudo-action and if* $f: B_1 \to B$ *is a continuous map, then*

$$(2)\qquad \mathscr{C}_i (f^* \xi) = f^* (\mathscr{C}_i (\xi)).$$

(*) Lavoro eseguito come professore visitatore del Consiglio Nazionale delle Ricerche presso l'Istituto Matematico « G. Castelnuovo » dell'Università di Roma.

(**) Nella seduta del 14 dicembre 1974.

(1) The present $\mathscr{C}_i$'s differ from the $\mathscr{C}_i$'s of [3] for what concerns the indices only.

(ii) *If* $S^1_B = (B \times S^1, pj_1, B, S^1, S^1)$ *with the natural* S^1*-pseudo-action, then for any fibre bundle with* S^1*-pseudo-action* ξ*:*

$$\mathcal{C}_i(\xi \oplus S^1_B) = \mathcal{C}_i(\xi).$$

(iii) *If* ξ_1 *and* ξ_2 *are two oriented fibre bundles with* S^1*-pseudo-action, then:*

$$\mathcal{C}_i(\xi_1 \oplus \xi_2) = \sum_{p+q=i} \mathcal{C}_p(\xi_1) \cup \mathcal{C}_q(\xi_2), \quad \textit{modulo odd torsion.} \tag{3}$$

(iv) *The Hopf fibration* $\theta_1 : S^3 \to S^2$ *is a fibre bundle with* S^1*-pseudo-action and*

$$\mathcal{C}_0(\theta_1) = 1 \quad , \quad \mathcal{C}_2(\theta_1) = c_1(\theta_1).$$

1.3. THEOREM B. *If* ξ *is the sphere bundle with* S^1*-pseudo-action associated with a complex vector bundle* η*, then:*

$$\mathcal{C}_{2i}(\xi) = c_i(\eta), \qquad 0 \leq i. \tag{4}$$

§ 2. PRELIMINARIES

2.1. Let $\Delta^p = [a_0, a_1, \cdots, a_p]$, be the standard simplex of dimension p. Our intent is to present a «triangulation» of the product $\Delta^p \times \Delta^q$. We introduce some notations:

$$\begin{aligned} J_{p,q} &= \left\{\varepsilon \mid \varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{p+q}), \quad \varepsilon_i = 0, 1, \quad \sum_i \varepsilon_i = p\right\} \\ |\varepsilon| &= \sum_{1 \leq i \leq p+q} (-1)^i \cdot i \cdot (1 - \varepsilon_i). \end{aligned} \tag{5}$$

The product of the linear structures in Δ^p and Δ^q is a linear structure in $\Delta^p \times \Delta^q$. Hence, if we take arbitrarily $(n+1)$ points $x_0, x_1, \cdots, x_n$ in $\Delta^p \times \Delta^q$, there exists an unique singular simplex $\sigma : \Delta^n \to \Delta^p \times \Delta^q$ such that σ is linear, and $\sigma(a_i) = x_i$, $0 \leq i \leq n$. We denote σ by $[x_0, x_1, \cdots, x_n]$.

If $\varepsilon \in J_{p,q}$ we define the $(p+q)$-singular simplex in $\Delta^p \times \Delta^q$:

$$\varepsilon(\Delta^p, \Delta^q) = [(a_0, a_0), (a_{\varepsilon_1}, a_{1-\varepsilon_1}), \cdots, (a_{\sum_{1 \leq i \leq r} \varepsilon_i}, a_{r - \sum_{1 \leq i \leq r} \varepsilon_i}), \cdots, (a_p, a_q)]. \tag{6}$$

With these notations we define the $(p+q)$-singular chain in $\Delta^p \times \Delta^q$, which we call the « triangulation » of $\Delta^p \times \Delta^q$:

$$\Delta^p \boxtimes \Delta^q = \sum_{\varepsilon \in J_{p,q}} (-1)^{pq + \frac{q(q+1)}{2}} |\varepsilon| \cdot \varepsilon(\Delta^p, \Delta^q). \tag{7}$$

A straighforward calculation proves the:

2.2. PROPOSITION I.

$$(7') \qquad \begin{aligned} &\text{(i)} \quad \Delta^0 \boxtimes \Delta^0 = [(a_0, a_0)];\\ &\text{(ii)} \quad \partial(\Delta^p \boxtimes \Delta^q) = (\partial\Delta^p) \boxtimes \Delta^q + (-1)^p \Delta^p \boxtimes (\partial\Delta^q);\\ &\text{(iii)} \quad \Delta^p \boxtimes (\Delta^q \boxtimes \Delta^r) = (\Delta^p \boxtimes \Delta^q) \boxtimes \Delta^r. \end{aligned}$$

Proposition 2.2. permits us to construct an explicit chain equivalence:

$$(8) \qquad \beta : C_*(X, \mathbf{Z}) \otimes C_*(Y, \mathbf{Z}) \to C_*(X \times Y, \mathbf{Z})$$

whose existence is asserted by the Eilenberg-Zilber theorem.

2.3. DEFINITION. *If* $\sigma : \Delta^p \to X$, $\mathcal{C} : \Delta^q \to Y$ *are two singular simplexes, we define*:

$$(9) \qquad \beta(\sigma \otimes \mathcal{C}) = (\sigma \times \mathcal{C})(\Delta^p \boxtimes \Delta^q).$$

2.4. COROLLARY.

$$(10) \quad \beta(\theta \otimes \beta(\sigma \otimes \mathcal{C})) = \beta(\beta(\theta \otimes \sigma) \otimes \mathcal{C}) \quad \text{for any } \theta \in C_*(X, \mathbf{Z}), \sigma \in C_*(Y, \mathbf{Z}), \mathcal{C} \in C_*(Z, \mathbf{Z}).$$

§ 3. PROOF OF THEOREM A AND THEOREM B

3.1. *Proof of Theorem A*. (0). See Definition 3.6 [3].

(i) See the proof of Theorem 7.13 (i) [2].

(ii) We construct a system of operators $k_p^{(2r)}$ (see [3]) for the bundle $\bar{\xi} = \Sigma\Sigma\,\xi$;

$$(11) \qquad \begin{aligned} &(1 + A)\, S k_p^{(2r-2)} = \partial k_p^{(2r)} - k_{p-1}^{(2r)} \partial_p, \qquad p \le n - 2r + 1,\\ &k_p^{(0)} = 1, \end{aligned}$$

where S is the homological S^1-pseudo-action defined by the S^1-pseudo-action in ξ.

We shall construct a system of operators $\vec{k}_p^{(2r)}$ defined in $\Sigma\Sigma\bar{\xi}$ such that:

–) $\vec{k}_p^{(2r)}$, is defined for $p \le n - 2r + 3$,

–) $\vec{k}_p^{(2r)}|_{\bar{\xi}} = k_p^{(2r)}$ for $p \le n - 2r + 1$,

–) $(1 + A)\, S\vec{k}_p^{(2r-2)} = \partial\vec{k}_p^{(2r)} - \vec{k}_{p-1}^{(2r)} \partial_p$,

S being also the homological S^1-pseudo-action induced by the S^1-pseudo-action in $\Sigma\Sigma\xi$,

–) if Σ denotes the suspension insomorphism in the homology of the fibres, then:

$$\Sigma\Sigma\hat{\omega}_{2r+2}(\Sigma\Sigma\bar{\xi}, \vec{k}_p^{(2s)})(\sigma) = \hat{\omega}_{2r}(\xi, \vec{k}_p^{(2s)})(\sigma)$$

for any singular simplex σ of dimension $n - 2r + 2$, $\hat{\omega}$ being the cochain defined in 7.2. [2].

Observe that the last equality proves the assertion (i).

We have by Definition 7.2 [2]:

$$(12) \qquad \hat{\omega}_{2r}(\bar{\xi}, k_p^{(2s)})(\sigma) = [((1 + A)\, S k_{n-2r-2}^{(2r-2)} + k_{n-2r+1}^{(2r)}\, \partial_{n-2r+2})\, \sigma].$$

The obstruction cocycle $\hat{\omega}_{2r+2}(\Sigma\Sigma\bar{\xi}, \tilde{k}_p^{(2s)})$ will be:

$$(13) \qquad \hat{\omega}_{2r+2}(\Sigma\Sigma\bar{\xi}, \tilde{k}_p^{(2s)})(\sigma) = [(1 + A)\, S\tilde{k}_{n-2r+2}^{(2r)} + \tilde{k}_{n-2r+1}^{(2r+2)}\, \partial_{n-2r+2})\, \sigma].$$

Let $C_{\pm}\bar{\xi} \subset \Sigma\bar{\xi} \subset \Sigma(\Sigma\bar{\xi})$ denote the cone bundles over $\bar{\xi}$, $(C_{-}\bar{\xi} = AC_{+}\bar{\xi}$, $C_{+}(\bar{\xi}) \cap C_{-}(\bar{\xi}) = \bar{\xi})$, and let $C_{\pm}(\Sigma\bar{\xi})$ have similar meanings.

The construction of $\tilde{k}_{n-2r+2}^{(2r)}$ in $\bar{\xi}$ meets an obstruction because it would be

$$\partial\tilde{k}_{n-2r+2}^{(2r)}(\sigma) = \omega_{2r}(\bar{\xi}, k_p^{(2s)})(\sigma) = ((1 + A)\, S k_{n-2r+2}^{(2r-2)} + k_{n-2r+1}^{(2r)}\, \partial)(\sigma);$$

the right hand side is a cycle but in generally is is not a boundary (in the homology complex of fibres). On the contrary, $\omega_{2r}(\bar{\xi}, k_p^{(2r)})(\sigma)$ is a boundary in the cone bundle $C_{+}\bar{\xi}$, and hence we can construct $\tilde{k}_{n-2r+2}^{(2r)}$ with values in the chain complex of $C_{+}\bar{\xi}$. The same argument permit us to construct $\tilde{k}_{n-2r+3}^{(2r)}$ with values in the chain complex of $C_{\pm}\Sigma\bar{\xi}$.

Now we shall analyse (13). We observe that

$$S(C_{+}\bar{\xi}) \subseteq C_{+}(\Sigma\bar{\xi}).$$

and hence $S\tilde{k}_{n-2r+2}^{(2r)}$ takes its values in the chain complex of $C_{+}(\Sigma\bar{\xi})$. The precedent construction of $\tilde{k}_{n-2r+1}^{(2r+2)}$ shows that it takes values in the same complex, while $A\tilde{k}_{n-2r+1}^{(2r+2)}$ takes values in the chain complex of $C_{-}(\Sigma\bar{\xi})$.

The suspension Σ acts on $\hat{\omega}_{2r+2}(\Sigma\Sigma\bar{\xi}, \tilde{k}_p^{(2r)})$ in such a way that:

$$\Sigma[\hat{\omega}_{2r}(\Sigma\Sigma\bar{\xi}, \tilde{k}_p^{(2r)})(\sigma)] = -[\partial(AS\tilde{k}_{n-2r+2}^{(2r)})(\sigma)]$$

where the last [ ] indicates the homology class of cycles on the homology of the fibres of $\Sigma\bar{\xi}$. But we have:

$$-AS\tilde{k}_{n-2r+2}^{(2r)}(\sigma) = ((1 - A) + AS\partial) \cdot \tilde{k}_{n-2r+2}^{(2r)}(\sigma).$$

We have observed that $\partial\tilde{k}_{n-2r+2}^{(2r)}(\sigma)$ is a chain in $\bar{\xi}$; hence $AS\partial\tilde{k}_{n-2r+2}^{(2r)}(\sigma)$ does also and in consequence $-\partial AS\tilde{k}_{n-2r+2}^{(2r)}(\sigma)$ can be decomposed in two chains: $\tilde{k}_{n-2r+2}^{(2r)}(\sigma)$ resp. $(-A + AS\partial)(\tilde{k}_{n-2r+2}^{(2r)}(\sigma)$ which lie in $C_{+}\bar{\xi}$, resp. $C_{-}\bar{\xi}$. We have in consequence

$$\Sigma\Sigma[\hat{\omega}_{2r}(\Sigma\Sigma\bar{\xi}, \tilde{k}_p^{(2r)}(\sigma)] = [\partial\tilde{k}_{n-2r+2}^{(2r)}(\sigma)] = \omega_{2r}(\bar{\xi}, \tilde{k}_p^{(2r)})(\sigma).$$

(iii) The proof follows principally the steps of the proof of Theorem 7.13. (iii) [2].

We consider the bundles $\bar{\xi}_i = \Sigma\Sigma\xi_i$, $i = 1, 2$, and let be $C\bar{\xi}_i$ their cone bundles.

The S^1-pseudo-actions in $\bar{\xi}_i$ extend naturally in $C\bar{\xi}_i$ and then the fibration

$$C\bar{\xi}_1 \times C\bar{\xi}_2 \to B_1 \times B_2 ,$$

(B_i is the base of ξ_i , $i = 1$, 2) has a S^1-pseudo-action.

We choose a system of local homomorfisms:

$$(14) \qquad \begin{cases} k_p^{(2r)} : C_* (\Sigma\Sigma\xi_1 , R) \to C_* (C\Sigma\Sigma\xi_1 , R) \\ h_p^{(2r)} : C_* (\Sigma\Sigma\xi_2 , R) \to C_* (C\Sigma\Sigma\xi_2 , R) \\ (1 + A)\, S k_p^{(2r-2)} = \partial k_p^{(2r)} - k_{p-1}^{(2r)} \partial_p , & k_p^{(0)} = 1 , \\ (1 + A)\, S h_q^{(2r-2)} = \partial h_q^{(2r)} - h_{q-1}^{(2r)} \partial_q , & h_q^{(0)} = 1 , \end{cases}$$

such that:

$$k_p^{(2r)} C_* (\Sigma\Sigma\xi_1 , R) \subset C_* (\Sigma\Sigma\xi_1 , R) , \qquad p \leq m - 2r + 1$$

$$h_q^{(2r)} C_* (\Sigma\Sigma\xi_2 , R) \subset C_* (\Sigma\Sigma\xi_2 , R) , \qquad q \leq n - 2r + 1 .$$

An easy calculation shows that if we define

$$(15) \qquad \begin{aligned} & H_{p,q}^{(r)} : C_p (\Sigma\Sigma\xi_1 , R) \otimes C_q (\Sigma\Sigma\xi_2 , R) \to \bigotimes_{p+q+r=i+j} C_i (\Sigma\Sigma\xi_1, R) \otimes C_j (\Sigma\Sigma\xi_2 , R) \\ & H_{p,q}^{(r)} = \sum_{r_1+r_2=r} (-1)^{pr_2} A^{r_2} k_p^{(r_1)} \otimes h_q^{(r_2)} , \end{aligned}$$

then:

$$(16) \qquad H_{p,q}^{(0)} = 1 \quad , \quad (1 + (-1)^{r+1} A)\, H_{p,q}^{(r-1)} = \partial H_{p,q}^{(r)} + (-1)^{r+1} H^{(r)} \partial_{p,q} .$$

A straightforward calculation shows also:

$$(17) \qquad H^{(1)} H^{(1)} = 0 \quad , \quad (1 + A)\, H^{(1)} H^{(2r-2)} = \partial H^{(2r)} - H^{(2r)} \partial .$$

Let

$$\alpha : C_* ((C\Sigma\Sigma\xi_1) \times (C\Sigma\Sigma\xi_2 , R) \to C_* (C\Sigma\Sigma\xi_1 , R) \otimes C_* (C\Sigma\Sigma\xi_2 , R)$$

denote the diagonal approximation (see e.g. E. Spanier–Algebraic Topology).

We define now a system of local homomorphisms

$$(18) \qquad \begin{aligned} & \tilde{H}^{(r)} : C_* ((C\Sigma\Sigma\xi_1) \times (C\Sigma\Sigma\xi_2 , R)) \circlearrowleft \\ & \tilde{H}^{(r)} = \beta H^{(r)} \alpha . \end{aligned}$$

The $\tilde{H}^{(r)}$'s satisfy the identities:

$$(1 + (-1)^{r+1} A)\, \tilde{H}^{(r-1)} = \partial \tilde{H}^{(r)} + (-1)^{r+1} \tilde{H}^{(r)} \partial ;$$

we have $\tilde{H}^{(0)} = \beta\alpha$ which is $\neq 1$ in general and also $\tilde{H}^{(2r+1)} \neq S\tilde{H}^{(2r)}$.

We intend to "correct" the $\tilde{H}^{(r)}$'s in such a way that the last two inequalities become equalities.

To this purpose we define in $C_* ((C\Sigma^2 \xi_1) \times (C\Sigma^2 \xi_2))$ two operators T and H of degree two and one, respectively.

We consider the product $\Delta^1 \times \Delta^1$ and we identify linearly Δ^2 with the triangle $[(0, 0), (1, 0), (1, 1)]$. If S', S'' represent the S^1-pseudo-actions in $C\Sigma^2 \xi_i$, $i = 1, 2$, we shall define a map:

$\mu : [(0, 0), (1, 0), (1, 1)] \times (C\Sigma^2 \xi_1) \times (C\Sigma^2 \xi_2) \to (C\Sigma^2 \xi_1) \times (C\Sigma^2 \xi_2)$ as follows:

if $(t_1, t_2) \in [(0, 0), (1, 0), (0, 1)]$ and $e_i \in C\Sigma^2 \xi_i$, $i = 1, 2$, then

$$\mu(t_1, t_2, e_1, e_2) = (S'(t_1, e_1), S''(t_2, e_2)). \tag{19}$$

If $\sigma : \Delta^r \to (C\Sigma^2 \xi_1) \times (C\Sigma^2 \xi_2)$ is a singular simplex, we define

$$T\sigma = \mu([(0, 0), (1, 0), (1, 1)] \boxtimes \Delta^r). \tag{20}$$

We have:

$$\partial T\sigma = \mu((\partial [(0, 0), (1, 0); (1, 1)]) \boxtimes \Delta^r + [(0, 0), (1, 0), (1, 1)] \boxtimes \partial\Delta^r). \tag{21}$$

We write by definition:

$$H\sigma = \mu([(0, 0), (1, 0)] + [(1, 0), (1, 1)]) \boxtimes \Delta^r); \tag{22}$$

We observe that:

$$S\sigma = \mu([(0, 0), (1, 1)] \boxtimes \Delta^r).$$

A straightforward calculation shows:

$$TA = AT \ , \ HA = AH \ , \ ST = TS \ , \ TH = HT \ , \ H\beta = \beta H, \tag{23}$$

and (21) becomes:

$$\partial T - T\partial = H - S. \tag{24}$$

Now we are able to get the desired corrections.

The relations (23) and (24) give us:

$$T(\partial T - T\partial) = (\partial T - T\partial) T \tag{25}$$

from which we deduce

$$(r + 1) T^r (\partial T - T\partial) = \sum_{0 \le s \le r} T^s (\partial T - T\partial) T^{r-s} = \partial T^{r+1} - T^{r+1} \partial, \tag{25'}$$

and hence:

$$T^r (\partial T - T\partial) = \frac{1}{r + 1} (\partial T^{r+1} - T^{r+1} \partial). \tag{25''}$$

Let $\bar{k}^{(2)} : C_*((C\Sigma^2 \xi_1) \times (C\Sigma^2 \xi_2), R) \circlearrowleft$ be a local system of operators which satisfy:

$$(1 + A) S = \partial \bar{k}^{(2)} - \bar{k}^{(2)} \partial .$$

If we take:

$$k^{(2)} = \tilde{H}^{(2)} + \bar{k}^{(2)} (1 - \beta\alpha) - (1 + A) T\beta\alpha, \tag{26}$$

we have also: $\partial k^{(2)} - k^{(2)} \partial = (1 + A) S$.

We define:

$$k^{(2r)} = \tilde{H}^{(2r)} + (1 + A) \sum_{1 \leq i \leq r} \frac{(-1)^i 2^{i-1}}{i!} T^i \tilde{H}^{(2r-2i)}, \qquad r \geq 2. \tag{27}$$

We remark that for any natural i, there exist natural numbers p_i, q_i, such that:

$$\frac{2^{i-1}}{i!} = \frac{p_i}{q_i}, \quad q_i = \text{odd}. \tag{28}$$

A direct calculation involving all properties of the operators considered above, shows that

$$(1 + A) S k^{(2r-2)} = \partial k^{(2r)} - k^{(2r)} \partial, \qquad r \geq 2. \tag{29}$$

Hence, we can use $k^{(2r)}$, $r \geq 1$, for the calculation of the classes $\mathscr{C}_i(\Sigma^2 \xi_1 \hat{\oplus} \xi_2)$.

We have:

$$\left(\prod_{1 \leq i \leq r} q_i\right) \omega_{2r}(\Sigma^2 \xi_1 \hat{\oplus} \xi_2, k^{(r)}) = \tag{30}$$

$$= \left[\left(\prod_{1 \leq i \leq r} q_i\right) \tilde{H}^{(2r)} + \left(\prod_{1 \leq i \leq r} q_i\right) \cdot \sum_{1 \leq i \leq r} \frac{(-1)^i}{i!} 2^{i-1} T^i \tilde{H}^{(2r-2i)}\right]$$

where [ ] denotes the local relative homology (see proof Theorem 7.13 (iii) [2]).

The term:

$$\left(\prod_{1 \leq i \leq r} q_i\right) \sum_{1 \leq i \leq r} \frac{(-1)^i}{i!} 2^{i-1} T^i \tilde{H}^{(2r-2i)}$$

is an integer chain and it is easily seen that its local relative homology class is zero as it lies in the subspace of the copy by respect to which we take the local relative homology. Therefore, modulo odd torsion, we obtain,

$$\omega_{2r}(\Sigma^4(\xi_1 \hat{\oplus} \xi_2), k^{(r)}) = [\tilde{H}^{(2r)}] = \sum_{s_1 + s_2 = 2r} (-1)^{ps_2} [A_1^{s_2} k^{(s_1)}_{m-s_1+2} \otimes h^{(s_2)}_{n-s_1+2}] = \tag{31}$$

$$= \sum_{s_1 + s_2 = 2r} (-1)^{ps} [A_1^s k^{(s)}_{m-s_1+2}] \otimes [h^{(s)}_{n-s_2+2}].$$

If s_1 or s_2 is odd, the relative cycle $k^{(s_1)}_{m-s_1+2}$ or $h^{(s_2)}_{n-s_2+2}$ is zero for the same argument; therefore:

$$\omega_{2r}(\Sigma^4(\xi_1 \hat{\oplus} \xi_2), k^{(r)}) = \sum_{r_1 + r_2 = r} [k^{(2r)}_{m-2r_1+2}] \otimes [h^{(2r)}_{n-2r_2+2}] \tag{32}$$

(modulo odd torsion)

from which we deduce:

$$\omega_{2r}(\Sigma^4(\xi_1 \hat{\oplus} \xi_2)) = \sum_{r_1 + r_2 = r} \omega_{2r_1}(\Sigma^2 \xi_1, k^{(r)}) \otimes \omega_{2r_2}(\Sigma^2 \xi_2, h^{(r)}) \tag{33}$$

(modulo odd torsion)

Using (ii), we deduce:

$$\mathscr{C}_{m+n-2r+4}(\xi_1 \hat{\oplus} \xi_2) = \sum_{r_1+r_2=r} \mathscr{C}_{m-2r_1+2}(\xi_1) \times \mathscr{C}_{n-2r_2+2}(\xi_2);$$

(modulo odd torsion)

from which we obtain the desired formula.

(iv) Using (ii), we have:

$$\mathscr{C}_0(\theta_1) = \omega_2(\theta_1, k_p^{(r)}) = 1.$$

For the verification of the second relation, we identify S^2 with $\dot{\Delta}^3$ and we construct a section

$$s : (\dot{\Delta}^3)^{(1)} \to S^3,$$

$(\dot{\Delta}^3)^{(1)}$ being the 1-skeleton of $\dot{\Delta}^3$; we can choose s such that s is extendible on 3 simplexes of dimension 2 of $\dot{\Delta}^{(3)}$; s cannot be extended over the remaining 2-simplex and the degree of s on this last simplex is 1. The section s will be used for the construction of $k_1^{(2)}$.

3.2. *Proof of Theorem B.*

The Chern classes are characterized exactly by the properties (o)–(iV) Theorem A (cfr. [1]). The classes $\mathscr{C}_i$ are natural and $H^*(BU, \mathbf{Z})$ is torsion free.

Remark the domain of definition of the Chern classes is more limited than the domain of definition of the $\mathscr{C}_i$'s.

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