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A Variant of Segal's construction of classifying spaces

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Topologia. — *A Variant of Segal's construction of classifying spaces.* Nota di GIUSEPPE ACCASCINA, presentata (*) dal Socio B. Segre.

RIASSUNTO. — Viene modificata la costruzione di spazio classificante (data da G. Segal in [5]) per una categoria topologica in modo tale che lo spazio classificante della categoria topologica associata ad un gruppo topologico sia omeomorfo allo spazio classificante definito da J. Milnor ([2]).

0. INTRODUCTION

Given a topological category $\mathscr{C}$, G. Segal ([5]) defines its classifying space to be the geometric realization of the semi-simplicial space given by the nerve of $\mathscr{C}$.

In this paper we regard the nerve of $\mathscr{C}$ only as a Δ-space (i.e. we do not consider the "degeneracy maps") and define the classifying space of $\mathscr{C}$ to be the geometric realization of such Δ-space. This construction looks more suitable than Segal's one because, when applied to the topological category associated to a topological group G, it gives a classifying space which is homeomorphic to the classifying space of G as defined by J. Milnor ([2]).

In the first section we define a Δ-space to be roughly a Δ-set ([4]) endowed with a topological structure. Milnor's geometric realization ([3]) adapts easily to define the realization T(A) of a Δ-space A. In our case T(A) is not necessarily a C. W. complex; this is due essentially to the presence of a non trivial topology in the spaces A^n of A. However one can prove that T(A) has a good filtration which gives a spectral sequence completely analogous to that of 5.1. in [5].

In the second section we make the definition of the nerve $N\mathscr{C}$ of a topological category $\mathscr{C}$ and prove that $N\mathscr{C}$ is a Δ-Space; then we define the classifying space of $\mathscr{C}$ to be $B\mathscr{C} = T(N\mathscr{C})$.

In the third section, given a topological group G, we first define two topological categories $\mathscr{G}$ and $\overline{\mathscr{G}}$, then we show that $B\overline{\mathscr{G}}$ is a free G-space and prove that $(B\overline{\mathscr{G}})_{/G} \cong B\mathscr{G}$.

Finally we show that the fibration $B\overline{\mathscr{G}} \to B\mathscr{G}$ is equivalent to the universal bundle $E \to B_G$ given in [2].

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1. Δ-SPACES AND GEOMETRIC REALIZATION

For any non negative integer p, let $[p]$ be the ordered set $\{0, 1, \cdots, p\}$. Let Δ be the category having as objects the sets $[p]$ and as morphisms the maps $\alpha : [p] \to [q]$ such that $\alpha(i) < \alpha(j)$ for every integer $0 \leq i < j \leq p$.

A Δ-space is a contravariant functor from the category Δ to the category Top of topological spaces and continuous maps. Since every morphism of Δ is the composition of maps $\partial_i : [p-1] \to [p]$ defined as follows: $\partial_i(j) = j$ for $j < i$, $\partial_i(j) = j + 1$ for $j \geq i$, a Δ-space A is a sequence of topological spaces $A^0, A^1, \cdots, A^n, \cdots, (A^n = A([n]))$ and continuous maps $\partial_i^* : A^n \to A^{n-1}$ $(i = 0, 1, \cdots, n)$ which verify the property $\partial_i^* \partial_j^* = \partial_{j-1}^* \partial_i^*$ if $i < j$.

Let Δ_p be the standard p-simplex in R^{p+1} every element $t \in \Delta_p$ is then given by $t = (t_0, \cdots, t_p)$ with $t_i \geq 0$ and $\sum_{i=0}^{p} t_i = 1$. Every map $\partial_i : [p-1] \to [p]$ induces a continuous map $\partial_{i*} : \Delta_{p-1} \to \Delta_p$ $\partial_{i*}(t_0, \cdots, t_{p-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{p-1})$, therefore any morphism of $\Delta \alpha : [p] \to [q]$ induces a continuous map $\alpha_* : \Delta_p \to \Delta_q$.

Let $A : \Delta \to \text{Top}$ be a Δ-space. The geometric realization of A is given by $T(A) = \left(\bigsqcup_{i=0}^{\infty} \Delta_i \times A^i \right) / \sim$ where $(t, \alpha^* a) \sim (\alpha_* t, a)$ where $t \in \Delta_p$, $a \in A^q$, $\alpha : [p] \to [q]$ is a morphism of Δ and $\bigsqcup$ is the disjoint union. Denote $q : \bigsqcup \Delta_i \times A^i \to T(A)$ the quotient map and $|t, a| \in T(A)$ the equivalence class of (t, a) and $T^p(A) = q\left(\bigsqcup_{i=0}^{p} \Delta_i \times A^i \right)$ and denote $q_p : \bigsqcup_{i=0}^{p} \Delta_i \times A^i \to T^p(A)$ the restriction of q.

Then we have the following:

PROPOSITION. *If* A *is a Δ-space, there is the following filtration of* T(A):

$$T^0(A) \subset T^1(A) \subset \cdots \subset T^p(A) \subset \cdots \subset T(A).$$

Proof. If C is a closed subspace of T(A) then for every p $C \cap T^p(A)$ is closed in $T^p(A)$ because $T^p(A)$ is a subspace of T(A). On the other way, if $C_p = C \cap T^p(A)$ is closed for every p then $q_p^{-1}(C_p)$ is closed for every p, therefore $q^{-1}(C)$ is closed and C is therefore closed because q is a quotient map.

We can now prove an analogue of the Theorem 5.1. of [5].

THEOREM. Let A *be a Δ-space and* $h_*(h^*)$ *and additive generalized homology (cohomology) theory, then there is a spectral sequence* $\{E^*_{p,q}\}$ $(\{E_*^{p,q}\})$ *which converges to* $H_*(T(A))$ $(H^*(T(A)))$ *with*:

$$E^2_{p,q} \cong \bar{H}_p(h_q(A)) \quad , \quad (E_2^{p,q} \cong \bar{H}^p(h^q(A)))$$

where $\bar{H}_p(h_q(A))$ *is the p-th homology group of the chain complex:*

$$h_q(A^0) \leftarrow \cdots \leftarrow h_q(A^p) \xleftarrow{d_{p+1}} h_q(A^{p+1}) \leftarrow \cdots$$

where $d_{p+1} = \sum_{i=0}^{n} (-1)^i h_q(\partial_i^*)$ *(and similarly for the cochain complex* $h^q(A)$*).*

Proof. Follow Segal's proof of 5.1. of [5] with the obvious modifications. For example, in our case we do not have degenerate parts.

2. Classifiyng Spaces of topological Categories

A topological category $\mathscr{C}$ is a category in which the set of objects $(\mathrm{Ob}\,\mathscr{C})$ and the set of morphisms $(\mathrm{Mor}\,\mathscr{C})$ have a structure of topological spaces and where the following maps are continuous:

$$\alpha : \mathrm{Mor}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{C} \quad , \quad \alpha(f) = \text{domain of } f;$$
$$\beta : \mathrm{Mor}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{C} \quad , \quad \beta(f) = \text{range of } f;$$
$$\gamma : \mathrm{Mor}\,\mathscr{C} \times \mathrm{Mor}\,\mathscr{C} \to \mathrm{Mor}\,\mathscr{C} \quad , \quad \gamma(f,g) = f \circ g \text{ (where the composition of maps makes sense).}$$

Given any topological category $\mathscr{C}$ we define the nerve of $\mathscr{C}$ to be the following Δ-space:

$$\mathrm{N}\mathscr{C} : \Delta \to \mathrm{Top} \quad , \quad [p] \to \mathrm{Funct}\,(p\,,\mathscr{C})$$

where p is the category having the set $[p]$ as set of objects and only a morphism from i to j whenever $i \leq j$. We have that $\mathrm{N}\mathscr{C}^p = \mathrm{N}\mathscr{C}([p])$ is a topological space for every p since it is $\mathrm{N}\mathscr{C}^0 = \mathrm{Ob}\,\mathscr{C}$, $\mathrm{N}\mathscr{C}^1 = \mathrm{mor}\,\mathscr{C}$ and in general $\mathrm{N}\mathscr{C}^p$ is given by all the sequences of p morphisms of $\mathscr{C}$:

$$\mathrm{C}_0 \xrightarrow{l_1} \mathrm{C}_1 \xrightarrow{l_2} \cdots \mathrm{C}_{p-1} \xrightarrow{l_p} \mathrm{C}_p \qquad \text{where} \quad \mathrm{C}_i \in \mathrm{Ob}\,\mathscr{C}$$

and therefore $\mathrm{N}\mathscr{C}^p$ has the topology induced by the topology of $\mathrm{Mor}\,\mathscr{C}$. It is easy to see that ∂_0^*, $\partial_1^* : \mathrm{N}\mathscr{C}^1 \to \mathrm{N}\mathscr{C}^0$ are respectively β and α. In general the map $\partial_i : [p - 1] \to [p]$ induces the following map $\partial_i^* : \mathrm{N}^p\mathscr{C} \to \mathrm{N}^{p-1}\mathscr{C}$

$$\partial_i^*(\mathrm{C}_0 \xrightarrow{l_1} \cdots \xrightarrow{l_{i-1}} \mathrm{C}_{i-1} \xrightarrow{l_i} \mathrm{C}_i \xrightarrow{l_{i+1}} \mathrm{C}_{i+1} \to \cdots \to \mathrm{C}_p) =$$
$$= \begin{cases} \mathrm{C}_0 \xrightarrow{l_1} \cdots \mathrm{C}_{i-1} \xrightarrow{l_{i+1} \circ l_i} \mathrm{C}_{i+1} \to \cdots \to \mathrm{C}_p & \text{if } 0 < i < p \\ \mathrm{C}_1 \xrightarrow{l_2} \mathrm{C}_2 \to \cdots \to \mathrm{C}_p & \text{if } i = 0 \\ \mathrm{C}_0 \xrightarrow{l_1} \mathrm{C}_1 \to \cdots \to \mathrm{C}_{p-1} & \text{if } i = p\,. \end{cases}$$

All these maps are continuous because of the properties stated in the definition of a topological category.

Given a topological category $\mathscr{C}$ we define its classifying space to be $\mathrm{B}\mathscr{C} = \mathrm{T}(\mathrm{N}\mathscr{C})$.

3. The Classifying Space of a Topological Group

Given any topological group G, following G. Segal, we define two topological categories $\mathscr{G}$, $\overline{\mathscr{G}}$. The category $\mathscr{G}$ has only one object and G as the space of morphisms, the product in G gives the composition law in $\mathscr{G}$; the category

$\overline{\mathscr{G}}$ has G as the space of objects and a unique morphism (g_0, g_1) for each ordered pair of elements (g_0, g_1) of G.

Since an element of $N^p\overline{\mathscr{G}}$ is a sequence $g = (g_0, g_1, \cdots, g_p)$ of $(p+1)$ elements of G we can define for every p a free action ρ of G over $N^p\overline{\mathscr{G}}$ by putting $\rho((g_0, \cdots, g_p)\bar{g}) = (g_0\bar{g}, \cdots, g_p\bar{g})$ where $\bar{g} \in G$. Since $\alpha^*\rho(g, \bar{g}) = \rho(\alpha^* g, \bar{g})$ for $g \in N^p\overline{\mathscr{G}}, \bar{g} \in G$ and $\alpha \in \operatorname{Mor}\Delta$, it is easily checked that the classifying space $B\overline{\mathscr{G}}$ is a free G-space.

For every p we have also a homeomorphism $k_p : (N^p\overline{\mathscr{G}})_{/G} \to N^p\mathscr{G}$ such that $k_q(\alpha^* g, g) = \alpha^* k_q(g, \bar{g})$ for every $\alpha : [p] \to [q]$ morphism of Δ, therefore $(B\overline{\mathscr{G}})_{/G}$ is homeomorphic to $B\mathscr{G}$.

Let us consider the construction of the universal boundle for a topological group G given by J. Milnor in [2]. Given a topological group G, the infinite join of G, $E = G * G * \cdots G * \cdots$, is defined to be the set $(t_0 g_0, \cdots, t_p g_p, \cdots)$ where t_i are real numbers such that $t_i \geq 0$, all but a finite number of t_i vanish and $\Sigma t_i = 1$, and $g_i \in G$ is given for any i such that $t_i \neq 0$. When $t_i = 0$ g_i can be chosen arbitrarly or omitted.

The following sets are a sub-basis of open sets of E:

1) the set of all $(t_0 g_0, \cdots, t_p g_p, \cdots)$ such that $\alpha < t_i < \beta$;

2) the set of all $(t_0 g_0, \cdots, t_p g_p, \cdots)$ such that $t_i \neq 0$, $g_i \in U_i$, U_i open in G.

Now define an action of G on R by putting:

$$\rho((t_0 g_0, \cdots, t_p g_p, \cdots)\bar{g}) = (t_0 g_0 \bar{g}, \cdots, t_p g_p \bar{g}).$$

The universal bundle is given by $E \to E_{/G} = B_G$.

THEOREM. *The two fibrations* $B\overline{\mathscr{G}} \to B\mathscr{G}$ *and* $E \to B_G$ *are equivalent.*

Proof. The following continuous maps:

$$h_p : \Delta_p \times N^p\overline{\mathscr{G}} \to G * G * \cdots * G \qquad (p+1)\text{-times}$$

given by $h_p(t; g) = h_p(t_0, \cdots, t_p; g_0, \cdots, g_p) = (t_0 g_0, \cdots, t_p g_p)$ verify the condition $h_q(\alpha_* t, g) = h_p(t, \alpha^* g)$, therefore they define a continuous map $h : B\overline{\mathscr{G}} \to E$.

Obviously h is onto. We have to prove that if $h|t', g'| = h|t, g|$ then $(t', g') \sim (t, g)$.

Let $(t', g') \in \Delta_p \times N^p\overline{\mathscr{G}}$ and $(t, g) \in \Delta_{p-r} \times N^{p-r}\overline{\mathscr{G}}$. We can assume $t_i \neq 0$ for every $i = 0, \cdots, p-r$; ortherwise if exist $i_1 < \cdots < i_s$ such that $t_{i_1} = \cdots = t_{i_s} = 0$ we have

$$(t, g) = (\partial_{i_1 *} \cdots \partial_{i_s *} t'', g) \sim (t'', \partial^*_{i_s} \cdots \partial^*_{i_1} g)$$

where

$$t'' = (t_1, \cdots, t_{i_1 - 1}, t_{i_1 + 1}, \cdots, t_p).$$

If $r = 0$ it must be $(t', g') = (t, g)$. If $r > 0$, since $t_i \neq 0$ for every i, from $h \mid t', g' \mid = h \mid t, g \mid$ it follows that exist $i_1 < \cdots < i_r$ such that

$$t' = (t_0, \cdots, t_{i_1-1}, 0, t_{i_1}, \cdots, t_{i_r-1}, 0, t_{i_r}, \cdots, t_{p-r})$$

and

$$g' = (g_0, \cdots, g_{i_1-1}, g'_{i_1}, \cdots, g_{i_{r-1}}, g'_{i_r}, \cdots, g_{p-r}).$$

Therefore if we put

$$t = (t_0, \cdots, t_{i_1-1}, t_{i_1}, \cdots, t_{p-r})$$

and

$$g = (g_0, \cdots, g_{i_1-1}, g_{i_1+1}, \cdots, g_{p-r})$$

we have

$$(t', g') = (\partial_{i_1*} \cdots \partial_{i_r*} t, g') \sim (t, \partial^*_{i_r} \cdots \partial^*_{i_1} g') = (t, g).$$

It is easy to see that h is a homeomorphism.

Moreover it commutes with the action of G and therefore induces an homeomorphism $h' : B\mathscr{G} \to B_G$ and the theorem is proved.

References

[1] G. Accascina (1970) – *Δ-Spaces and Classifying Spaces*, September 1970. University of Warwick.

[2] J. Milnor (1956) – *Construction of Universal Bundles*, II, «Ann. of Math.», *63*, 430–436.

[3] J. Milnor (1957) – *The Geometric Realization of a Semi-Simplicial Complex*, «Ann. of Math.», *65*, 357–362.

[4] C. P. Rourke and B. J. Sanderson (1971) – *Δ-Sets*. I: Homotopy Theory «Quart. J. Math.», Oxford, ser. II, *22*, 321–338.

[5] G. Segal (1968) – *Classifying Spaces and Spectral Sequences*, «Publ. Math. Ist. des Hautes Etudes Scient. (Paris)», *34*, 105–112.