Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On special block designs

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.6, p. 592–595.

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Geometrie finite. — *On special block designs.* Nota di D. M. MADURAM, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — R. H. Schulz ha dimostrato che ogni Block-plane con un gruppo di traslazione transitivo abeliano, che abbia un uguale numero di punti su ogni sua retta, determina una quasi–fibrazione [2]. Qui caratterizziamo le classi di isomorfi di tali Block-plane in relazione alle quasi-fibrazioni e definiamo una loro matrice rappresentativa.

I. INTRODUCTION

R. H. Schulz proved that every block design with a transitive abelian translation group, which has equal number of points on its lines determines a quasi-congruence [2]. We characterise here the isomorphism classes of these special block designs with respect to quasi-congruences and define a matrix representation of them.

II. SPECIAL BLOCK DESIGNS

DEFINITION. *A quasi-congruence* S *of the vector space* V_{mn} *of dimension* mn *over the Galois field* $F = GF(q)$ *is a set of n–dimensional subspaces, which are called quasi–components, such that each non–zero element of* V_{mn} *belongs to exactly* $\lambda \geq 1$ *quasi–components.*

THEOREM. *Any translational block design which has equal number of points on its lines and whose translation group is abelian can be considered as a block design of the form* $D(V_{mn}, S, F)$ *where points are the vectors of* V_{mn} *and the blocks are the quasi-components of* S *and their translates under the group of translates of* V_{mn}.

PROPOSITION 1. *The block designs* $D(V_{mn}, S, F)$ *and* $D(V_{mn}, S', F)$ *corresponding to quasicongruences* S *and* S′ *are isomorphic if and only if there exists a non-singular semilinear transformation of the vector space* V_{mn} *mapping each quasi-components of* S *onto that of* S′.

Proof. Let $D(V_{mn}, S, F)$ and $D(V_{mn}, S', F)$ be isomorphic. Then there exists an isomorphism f from $D(V_{mn}, S, F)$ onto $D(V_{mn}, S', F)$. Since the block designs have transitive abelian translation group, we assume without loss of generality that f maps the origin of V_{mn} onto itself i.e. $f(0) = 0$.

For any non-zero vector v in V_{mn}, let $\boldsymbol{v}$ denote the translation in V_{mn} given by $x \rightarrow x + v$. This induces a translation of the block design $D(V_{mn}, S, F)$. Now consider the mapping $f \cdot \boldsymbol{v} \cdot f^{-1}$ in $D(V_{mn}, S, F)$. Clearly this maps the

(*) Nella seduta del 14 novembre 1974.

blocks of $D(V_{mn}, S, F)$ onto the blocks of $D(V_{mn}, S, F)$ and we prove that this mapping is indeed a translation of $D(V_{mn}, S, F)$. First we note that this collineation is fixed point free; for if $f \cdot \boldsymbol{v} \cdot f^{-1}(X) = X$ for some point X, then $\boldsymbol{v} \cdot (f^{-1}(X)) = f^{-1}(X)$, contrary to the translation $\boldsymbol{v}$ being fixed point free. Now we need only to show that $f \cdot \boldsymbol{v} \cdot f^{-1}$ is a dilatation. If the points X and $f \cdot \boldsymbol{v} \cdot f^{-1}(X)$ belong to the same block b, then $f^{-1}(X)$ and $\boldsymbol{v} \cdot (f^{-1}(X))$ belong to the block $f^{-1}(b)$. Since any translation is again a dilatation, the block $f^{-1}(b)$ is fixed. Hence $\boldsymbol{v} \cdot (f^{-1}(b)) = f^{-1}(b)$ and so we have $f \cdot \boldsymbol{v} \cdot f^{-1}(b) = b$. Thus the collineation $f \cdot \boldsymbol{v} \cdot f^{-1}$ fixes the block b and so it is a dilatation.

Since $f \cdot \boldsymbol{v} \cdot f^{-1}$ is a translation and $f \cdot \boldsymbol{v} \cdot f^{-1}(o) = f(v)$, we have $f \cdot \boldsymbol{v} \cdot f^{-1}(X) = f(v) + X$. Hence

$$f(v_1 + v_2) = f \cdot \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \cdot f^{-1}(o) = f \cdot \boldsymbol{v}_1 \cdot f^{-1} \cdot f \cdot \boldsymbol{v}_2 \cdot f^{-1}(o) =$$
$$= f \cdot \boldsymbol{v}_1 \cdot f^{-1}(f(v_2)) = f(v_1) + f(v_2). \qquad \text{Thus } f \text{ is linear.}$$

Since each non-zero element of the basic field can be considered as a central dilatation of $D(V_{mn}, S, F)$, we have for each element k of F, $k \cdot v = \boldsymbol{k} \cdot \boldsymbol{v} \cdot \boldsymbol{k}^{-1}(o)$, where $\boldsymbol{k}$ denotes the dilatation induced by the left multiplication of the vectors of V_{mn} by k in $D(V_{mn}, S, F)$. Also $f \cdot \boldsymbol{k} \cdot f^{-1}$ induces a dilatation on $D(V_{mn}, S, F)$ fixing each of the quasi-components of S and so it represents a non-zero element of the field, say k^α in F. If we further let $o^\alpha = o$, then the mapping α is indeed an automorphism of the field F. In fact, we have for any v in V_{mn}; $f \cdot (\boldsymbol{k}_1 + \boldsymbol{k}_2) \cdot f^{-1}(v) = f \cdot \boldsymbol{k}_1 \cdot f^{-1} + f \cdot \boldsymbol{k}_2 \cdot f^{-1}(v)$ and so $(k_1 + k_2)^\alpha = k_1^\alpha + k_2^\alpha$ and similarly $(k_1 \cdot k_2)^\alpha = k_1^\alpha \cdot k_2^\alpha$.

Finally,

$$f(k \cdot v) = f \cdot \boldsymbol{k} \cdot \boldsymbol{v} \cdot \boldsymbol{k}^{-1}(o) = f \cdot \boldsymbol{k} \cdot \boldsymbol{v} \cdot \boldsymbol{k}^{-1} \cdot f^{-1}(o) =$$
$$= f \cdot \boldsymbol{k} \cdot f^{-1} \cdot f \cdot \boldsymbol{v} \cdot f^{-1} \cdot f \cdot \boldsymbol{k}^{-1} \cdot f^{-1}(o) = \boldsymbol{k}^\alpha \cdot f \cdot \boldsymbol{v} \cdot f^{-1} \cdot \boldsymbol{k}^{-1\alpha}(o) =$$
$$= \boldsymbol{k}^\alpha \cdot f(v) = k^\alpha \cdot f(v).$$

Since f is also one to one, it is a non-singular semi-linear transformation of the vector space V_{mn}.

Converse follows directly. The above proof also yields the following result:

PROPOSITION 2. *Every collineation fixing a point of the block design* $D(V_{mn}, S, F)$ *is induced by a non-singular semilinear transformation of the-vectorspace* V_{mn} *preserving the quasi-congruence* S.

PROPOSITION 3. *Suppose* $D_1, D_2, \cdots, D_k$ *are the totality of non-isomorphic block designs containing* p^{mn} *points, where* p *is a prime number and having equal number of points on each line and having transitive abelian translation group, then*

$$N/(p^{mn} - 1)(p^{mn} - p) \cdots (p^{mn} - p^{mn-1}) = 1/h_1 + 1/h_2 + \cdots 1/h_k ;$$

where N *is the total number of distinct quasi-congruences in* V_{mn} *over* GF(p) *and* h_i *is the order of the collineation subgroup fixing a point of the block design* D_i.

Proof. By Proposition 2, the number of distinct quasi-congruences representing isomorphic block designs $D(V_{mn}, S, F)$ is equal to the index of the group of non-singular linear transformations fixing the quasi-congruence S in the full group of all non-singular transformations of V_{mn}. Hence the result.

III. Matrix representation of block designs

We first consider the matrix representation of the lines through origin of the block designs $D(V_{mn}, S, F)$.

PROPOSITION 4. *To the lines through origin of the block design* $D(V_{mn}, S, F)$ *corresponds a set of rows of* $i \times i$ *square matrices* $(A_1^p, A_2^p, \cdots, A_k^p)$ *containing also unit row k-tuples* $(1, 0, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1)$ *with* $mn = ik$, *where i is some fixed divisor of n; such that for any two k-tuples of theform* A^p *and* A^q,

$$Rank \begin{pmatrix} A_1^p & A_2^p & \cdots & A_k^p \\ A_1^q & A_2^q & \cdots & A_k^q \end{pmatrix} = 2i .$$

Further, any quasi-component can be represented by a set of $n/i = c$ *lines* $A^1, A^2, \cdots, A^c$ *such that*

$$Rank \begin{pmatrix} A_1^1 & A_2^1 & \cdots & A_k^1 \\ A_1^2 & A_2^2 & \cdots & A_k^2 \\ \cdots & \cdots & \cdots & \cdots \\ A_1^c & A_2^c & \cdots & A_k^c \end{pmatrix} = n.$$

Proof. The lines through origin of the block design $D(V_{mn}, S, F)$ determine mutually disjoint (except for the zero vector) i-dimensional subspaces covering each of the quasi-components of the quasi-congruence S. Now i divides n. Let $mn = ik$. We choose a basis of mn vectors $e_1, e_2, \cdots, e_{mn}$ where the first i vectors lie in one line, the second subset of i vectors lie in another line and so on. Now each vector $x_1 e_1 + \cdots + x_{mn} e_{mn}$ can be expressed as an ordered k-tuple $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k)$ where each $\boldsymbol{x}_p = (x_{pi+1}, \cdots, x_{pi+i})$ and our choice of the first k fundamental lines given by the unit row k-tuples. For any other line, let us choose a basis on i k-tuples, $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k), \cdots, (\boldsymbol{z}_1, \cdots, \boldsymbol{z}_k)$. Now

$$\begin{pmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{z}_1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{z}_2 \end{pmatrix}, \cdots, \begin{pmatrix} \boldsymbol{x}_k \\ \vdots \\ \boldsymbol{z}_k \end{pmatrix}$$

are all $i \times i$ matrices and we denote this ordered set of matrices as A^i. Now the first condition of the proposition follows from the fact that any pair of distinct lines have only zero in common. Since each quasi-components contains exactly n independent vectors, we get the second condition also.

Note: If $i = n$, then the block design is just a translation plane and we get the usual matrix representation by taking $(A_1^r)^{-1}A_2^r$ for $A_1^r \neq 0$.

If, in each fundamental line, we change the basic vectors $e_{pi+1}, \cdots, e_{pi+i}$ to $e'_{pi+1}, \cdots, e'_{pi+i}$ given by a non-singular matrix C_i, then each line represented by A^r has a new representation of the form $A^r C_i$. Further choosing a different basis of this line given by a non-singular matrix B_r, we get the most general representation of the same line as $B_r A^r C_i$. Thus we have:

PROPOSITION 5. *If a set of k-tuples of* $i \times i$ *matrices*, $(A_1^r, A_2^r, \cdots, A_k^r)$ *correspond to the lines through origin of the block design* $D(V_{mn}, S, F)$, *then for arbitrary choices of fixed non-singular matrices* $C_1, C_2, \cdots, C_k$ *and* B_r: *the set of k-tuples* $(B_r A_1^r C_1, B_r A_2^r C_2, \cdots, B_r A_k^r C_k)$ *represent lines of an isomorphic block design.*

Acknowledgement. The Author wishes to thank his supervisor, Prof. M. Venkataraman, for his helpful guidance.

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