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Hyperasymptotic and hypergeodesic curvatures of a curve in special Kawaguchi spaces

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Geometria differenziale. — *Hyperasymptotic and hypergeodesic curvatures of a curve in special Kawaguchi spaces.* Nota di UDAI PRATAP SINGH e SHRI KRISHNA DEO DUBEY, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Studio delle curvature iperasintottiche e ipergeodetiche di una curva appartenente ad uno spazio di Kawaguchi di ordine due.

1. INTRODUCTION

Consider an n-dimensional special Kawaguchi space of order 2 such that the arc length of a curve $x^i = x^i(t)$ (1) is given by the integral

$$(1.1) \qquad S = \int [A_i(x, \overset{\prime}{x}) \overset{\prime\prime}{x}{}^i + B(x, \overset{\prime}{x})]^{1/p} \, dt , \qquad p \neq 0, 3/2$$

where $\overset{\prime}{x}{}^i = dx^i/dt$, $\overset{\prime\prime}{x}{}^i = d^2x^i/dt^2$ and A_i, B are differentiable functions of x^i and $\overset{\prime}{x}{}^i$. In order that the arc length be related intrinsically to the curve that is, it remains unaltered by a transformation of the parameter t, we must have (Kawaguchi [1] (2))

$$(1.2) \qquad A_i \overset{\prime}{x}{}^i = 0 ,$$

$$(1.3) \qquad 2 A_i \overset{\prime\prime}{x}{}^i + (A_{k(i)} \overset{\prime\prime}{x}{}^k + B_{(i)}) \overset{\prime}{x}{}^i = p (A_i \overset{\prime\prime}{x}{}^i + B) ,$$

where

$$A_{k(i)} = \partial A_k / \partial \overset{\prime}{x}{}^i \quad , \quad B_{(i)} = \partial B / \partial \overset{\prime}{x}{}^i .$$

Equation (1.3) implies

$$(1.4) \qquad A_{k(i)} \overset{\prime}{x}{}^i = (p - 2) A_k \quad , \quad B_{(i)} \overset{\prime}{x}{}^i = p B .$$

Thus A_i are homogeneous of degree $p - 2$ with regard to the $\overset{\prime}{x}{}^i$ and B is homogeneous of degree p.

We consider an m-dimensional subspace K_m of K_n represented as $x^i = x^i(u^\alpha)$ and the matrix of the projection factor $p^i_\alpha = \partial x^i / \partial u^\alpha$ has rank m. If we denote by a_α and b the quantities in K_m corresponding to A_i and B in K_n

(*) Nella seduta del 14 dicembre 1974.

(1) Latin indices run from 1 to n, Greek ones $\alpha, \beta, \gamma, \delta, \varepsilon, \rho$ from 1 to m and μ, ν from $m + 1$ to n.

(2) Numbers in the brackets refer to the references at the end of the paper.

then it follows that the equations similar to (1.2), (1.3) and (1.4) hold for a_α and b. Putting

$$G_{ij} \overset{\text{def}}{=} 2A_{i(j)} - A_{j(i)} \quad , \quad G_{\alpha\beta} \overset{\text{def}}{=} 2\, a_{\alpha(\beta)} - a_{\beta(\alpha)} \, ,$$

it has been shown (Yoshida [3]) that

$$G_{ij}\, p^i_\alpha\, p^j_\beta = G_{\alpha\beta} \, . \tag{1.5}$$

The covariant differential of a contravariant vector field $v^i(x^i, \acute{x}^i)$ homogeneous of degree zero with respect to $\acute{x}^i$ is defined as (Kawaguchi [1])

$$\delta v^i = dv^i + \Gamma^i_{jk}\, v^j\, dx^k \, , \tag{1.6}$$

where

$$2\Gamma^i = (2A_{lm}\, \acute{x}^m - B_{(l)})\, G^{li} \, ,$$

$$\Gamma^i_{jk} = \partial^2 \Gamma^i / \partial \acute{x}^j\, \partial \acute{x}^k \quad , \quad A_{lm} = \partial A_l / \partial x^m \, .$$

If v^α be a vector field in K_m such that $v^i = p^i_\alpha\, v^\alpha$, then the induced covariant differential $\check{\delta} v^\alpha \ (= p^\alpha_i\, \delta v^i)$ is given by (Yoshida [3])

$$\check{\delta} v^\alpha = dv^\alpha + \check{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, du^\gamma \, , \tag{1.7}$$

where

$$\check{\Gamma}^\alpha_{\beta\gamma} = p^\alpha_i\, (p^i_{\beta\gamma} + \Gamma^i_{jk}\, p^j_\beta\, p^k_\gamma) \, , \tag{1.8}$$
$$p^\alpha_i = G^{\alpha\beta}\, G_{ij}\, p^j_\beta \quad , \quad p^i_{\beta\gamma} = \partial p^i_\beta / \partial u^\gamma \, .$$

Further, Yoshida ([3]) has defined

$$\overset{\circ}{H}{}^i_{\beta\alpha} \overset{\text{def}}{=} \overset{\circ}{D}_\beta\, p^i_\alpha \overset{\text{def}}{=} p^i_{\alpha\beta} + \Gamma^i_{jk}\, p^j_\beta\, p^k_\alpha - \tilde{\Gamma}^\gamma_{\alpha\beta}\, p^i_\gamma \tag{1.9}$$

and expressed

$$\overset{\circ}{H}{}^i_{\beta\alpha} = \sum_\mu \underset{\mu}{H}_{\beta\alpha}\, \underset{\mu}{n}^i \, , \tag{1.10}$$

in which $\underset{\mu}{n}^i$ are vectors normal to K_m and $\underset{\mu}{H}_{\beta\alpha}$ are second fundamental tensors.

For a curve $C : x^i = x^i(s)$ of the subspace K_m, it has been shown (Yoshida [3]) that

$$q^i - p^i_\alpha\, p^\alpha = \overset{\circ}{H}{}^i_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = \sum_\mu \underset{\mu}{H}_{\beta\gamma}\, \underset{\mu}{n}^i \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \, , \tag{1.11}$$

where

$$q^i = \frac{\delta}{ds}\left(\frac{dx^i}{ds}\right) \quad , \quad p^\alpha = \frac{\check{\delta}}{ds}\left(\frac{du^\alpha}{ds}\right) . \tag{1.12}$$

It has been proved that the necessary and sufficient condition that the curve be an asymptotic line is that

$$\underset{\mu}{H}_{\beta\gamma} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} = 0 , \qquad \text{for} \quad \mu = m + 1 , \cdots , n. \tag{1.13}$$

2. Hyperasymptotic curvature

Consider a congruence of curves on K_m given by the vector field λ^i. At a point of the subspace this can be expressed as

$$\lambda^i = t^{\alpha} p^i_{\alpha} + \sum_{\mu} \Gamma_{(\mu)} \underset{\mu}{n}^i . \tag{2.1}$$

Let this vector be normalized by the condition

$$G_{ij}(x, \dot{x}) \lambda^i \lambda^j = 1 , \tag{2.2}$$

which gives

$$G_{\alpha\beta}(u, \dot{u}) t^{\alpha} t^{\beta} + \sum_{\mu} \Gamma^2_{(\mu)} \underset{\mu}{\psi} = 1 , \tag{2.3}$$

where

$$\underset{\mu}{\psi} = G_{ij}(x, \dot{x}) \underset{\mu}{n}^i \underset{\mu}{n}^j . \tag{2.4}$$

Let $b^i_{(0)} (\equiv dx^i/ds)$, $b^i_{(1)}$ and $b^i_{(\varphi)}$ $(\varphi = 2, \cdots, n-1)$ be the unit tangent, unit principal normal and $(n-2)$ unit binormal vectors of a curve $C : u^{\alpha} = u^{\alpha}(s)$ (of K_m) which is not an autoparallel curve in K_n.

Definition 2.1. A curve (of K_m) is said to be hyperasymptotic curve (of order $(\varphi - 1)$) of the subspace relative to λ^i if the surface determined by $b^i_{(0)}$ and the binormal vector $b^i_{(\varphi)}$ contains the vector field λ^i. In other words, we have (Singh [4])

$$\lambda^i = u b^i_{(0)} + v_{(\varphi)} b^i_{(\varphi)} , \qquad (\varphi = 2 \text{ or } 3 \text{ or } \cdots \text{ or } n-1) . \tag{2.5}$$

On comparing the equation (2.5) with the equation (2.1), we get

$$t^{\alpha} p^i_{\alpha} + \sum_{\mu} \Gamma_{(\mu)} \underset{\mu}{n}^i = u b^i_{(0)} + v_{(\varphi)} b^i_{(\varphi)} . \tag{2.6}$$

After multiplying (2.6) by $G_{ij} q^j$ and using the facts

$$G_{ij} b^i_{(0)} q^j = 0 \quad , \quad G_{ij} b^i_{(\varphi)} q^j = 0 , \tag{2.7}$$

we obtain

$$G_{ij} q^j t^{\alpha} p^i_{\alpha} + \sum_{\mu} \Gamma_{(\mu)} G_{ij} q^j \underset{\mu}{n}^i = 0 . \tag{2.8}$$

Using the equations (1.5), (1.11), (2.4), (2.8) and $G_{ij}\, p^i_\alpha \underset{\mu}{n}^j = 0$, we get

$$G_{\alpha\beta}\, t^\alpha p^\beta + \sum_\mu \Gamma_{(\mu)} \underset{\mu}{\psi} \underset{\mu}{H}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0 , \tag{2.9}$$

which represents hyperasymptotic curve relative to λ^i. The equation (2.5) and (2.7) give

$$G_{ij}(x , \acute{x})\, \lambda^i q^j = 0 , \tag{2.10}$$

therefore we have the following

THEOREM 2.1. *For a hyperasymptotic curve, the first curvature vector q^i is normal to λ^i.*

DEFINITION 2.2. The scalar K^* is defined by

$$K^* = G_{ij}(x , \acute{x})\, \lambda^i (u , \acute{u})\, q^j \tag{2.11}$$

is called the hyperasymptotic curvature (of a curve) relative to $\lambda^i(u , \acute{u})$.

It is obvious that the hyperasymptotic curvature of a curve vanishes (Prasad [2]) if and only if it is an hyperasymptotic line.

After using equations (1.11) and (2.1), we obtain

$$K^* \equiv G_{\alpha\beta}\, t^\alpha p^\beta + \sum_\mu \Gamma_{(\mu)} \underset{\mu}{\psi} \underset{\mu}{H}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} , \tag{2.12}$$

which yields

THEOREM 2.2. *If a hyperasymptotic line is an asymptotic line then either* (*i*) *the congruence is normal to* K_m *or* (ii) *the first curvature vector of the curve with respect to* K_m *is orthogonal to the component of the congruence tangential to* K_m.

Proof. If the hyperasymptotic line is asymptotic, then using equations (1.13), (2.12) and the fact that $K^* = 0$ we get

$$G_{\alpha\beta}\, t^\alpha p^\beta = 0. \tag{2.13}$$

Since the curve is asymptotic and not an autoparallel curve in K_m, $p^\alpha \neq 0$.

Therefore, equation (2.13) implies that either (i) $t^\alpha = 0$ or (ii) p^α is orthogonal to t^α. This proves the theorem.

3. HYPERGEODESIC CURVATURE

For a curve $C : u^\alpha = u^\alpha(s)$ of the subspace K_m, the vector with the components

$$\eta^\alpha = p^\alpha - \left\{ \sum_\mu \left(\underset{\mu}{H}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\mu \Gamma^2_{(\mu)} \right\}^{1/2} \left(t^\alpha - \lambda \frac{du^\alpha}{ds} \right) \tag{3.1}$$

is called the union curvature vector relative to λ^i (Singh and Dubey [5]) where $\lambda \overset{\text{def}}{=} G_{ij} \frac{dx^i}{ds} \lambda^j$.

The scalar k_u defined by

$$k_u^2 = G_{\alpha\beta}(u, \acute{u})\, \eta^\alpha \eta^\beta$$

is called the union curvature of the curve and this can be expressed as

$$k_u^2 = k_g^2 - 2 \left\{ \sum_\mu \left(\underset{\mu}{H}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\mu \Gamma^2_{(\mu)} \right\}^{1/2} \times \tag{3.2}$$
$$\times\, t_\rho\, p^\rho + \left\{ \sum_\mu \left(\underset{\mu}{H}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\mu \Gamma^2_{(\mu)} \right\} (G_{\alpha\rho}\, t^\alpha t^\rho - \lambda^2),$$

in which we have used the relations

$$G_{\alpha\beta}(u, \acute{u}) \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 1 \quad , \quad G_{\alpha\beta}(u, \acute{u})\, p^\alpha \frac{du^\beta}{ds} = 0 \tag{3.3}$$

and $k_g^2 = G_{\alpha\beta}(u, \acute{u})\, p^\alpha p^\beta$ (k_g is the first curvature of the curve).

From equations (1.13) and (3.2), we have

THEOREM 3.1. *The union and first curvatures are identical along an asymptotic line of the subspace.*

Let us suppose that $l^\alpha \overset{\text{def}}{=} p^\alpha / k_g$, multiplying (3.1) by $G_{\alpha\beta}(u, \acute{u})\, l^\beta$ and using the fact that

$$G_{\alpha\beta}(u, \acute{u}) \frac{du^\alpha}{ds} p^\beta = 0,$$

we get

$$G_{\alpha\beta}(u, \acute{u})\, \eta^\alpha l^\beta = k_g - \left\{ \sum_\mu \left(\underset{\mu}{H}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\mu \Gamma^2_{(\mu)} \right\}^{1/2} (G_{\alpha\beta}(u, \acute{u})\, t^\alpha l^\beta). \tag{3.4}$$

DEFINITION. The scalar K_g^* defined by

$$K_g^* = k_g - \left\{ \sum_\mu \left(\underset{\mu}{H}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\mu \Gamma^2_{(\mu)} \right\}^{1/2} (G_{\alpha\beta}(u, \acute{u})\, t^\alpha l^\beta) \tag{3.5}$$

is called the hypergeodesic curvature of the curve in K_m.

If K_g^* vanishes along a curve in K_m, then the curve is called a hypergeodesic. Therefore, a hypergeodesic is given by

$$G_{\alpha\beta}(u, \acute{u})\, \eta^\alpha l^\beta = 0. \tag{3.6}$$

This equation yields the following theorems:

THEOREM 3.2. *A union curve is a hypergeodesic but the converse is not necessarily true.*

THEOREM 3.3. *A non-union hypergeodesic curve is characterised by the property that its union curvature vector is orthogonal to the first curvature vector in* K_m.

Also, we have

THEOREM 3.4. *The hypergeodesic and first curvatures of a curve are equal if and only if either* (*i*) *the curve is the asymptotic line or* (*ii*) *the congruence* λ^i *is normal to* K_m.

The proof is obvious from equation (3.5).

Theorem 3.1 and Theorem 3.2 yield

THEOREM 3.5. *The hypergeodesic and union curvatures of an asymptotic line are identical, each being equal to the first curvature.*

Finally, we conclude with the following:

THEOREM 3.6. *Along a asymptotic line of the subspace union, first and hypergeodesic curvatures are identical.*

Remark. Since the curve under consideration is not an autoparallel curve of the embedding space, the above theorem reveals that an asymptotic line can not be a union curve or a hypergeodesic.

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