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The special infinitesimal projective transformation in an n-dimensional special Kawaguchi space

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Geometria differenziale. — *The special infinitesimal projective transformation in an n–dimensional special Kawaguchi space.* Nota di H. D. PANDE e B. SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Negli spazi speciali di Kawaguchi questi ha definito trasformazioni proiettive: gli Autori ne studiano le trasformazioni infinitesime.

1. INTRODUCTION

A Kawaguchi [1] (1) has defined a space with metric function $F = A_i x''^i + B$, such that $ds = F^{1/p} dt$. In view of this special type of metric and of the special case $p \neq 3/2$, 0; the space is called an n-dimensional special Kawaguchi space $K_n (n > 2)$. Here A_i and B are differentiable homogeneous functions of degrees $p - 2$ and p respectively. The projective transformation in a special Kawaguchi space has been considered by S. Kawaguchi [3] and different types of curvature tensors have been introduced. Here in this paper we study the special infinitesimal projective transformation in K_n and establish certain theorems. We shall use some of the ideas, notations and results given by A. Kawaguchi [2] without explanation.

We consider a contravariant vector field $X^i(x, x')$ which is homogeneous of degree zero with respect to x'^i. The covariant derivative of X^i is defined by

$$\nabla_j X^i = \partial_j X^i - \partial'_k X^i \Gamma^k_{(j)} + \Gamma^i_{(j)(k)} X^k, \tag{1.1}$$

denoting $\partial_j = \partial/\partial x^j$ and $\partial'_j = \partial/\partial x'^j$. The connection parameter $\Gamma^i(x, x')$ which is positively homogeneous of degree two in x'^i is given by

$$2\Gamma^i = (2 A_{lk} x'^k - B_{(l)}) G^{li}, \tag{1.2}$$

$$G_{ik} = 2 A_{i(k)} - A_{k(i)},$$

where

$$A_{k(i)} = \partial A_k/\partial x'^i \quad , \quad A_{ik} = \partial A_i/\partial x^k \quad , \quad B_{(i)} = \partial B/\partial x'^i .$$

The differential equation of path in K_n is

$$x^{[2]i} = x''^i + 2\Gamma^i = 0 .$$

(*) Nella seduta del 14 dicembre 1974.

(1) Numbers in brackets refer to the references at the end of the paper.

Using the above covariant derivative of vector X^i, the following curvature tensors have been obtained:

$$(1.3)\qquad a)\quad (\nabla_j \nabla_k - \nabla_k \nabla_j)\, X^i = - R_{jkl}^{\cdot\cdot\cdot i}\, X^l + K_{jk}^{\cdot\cdot l}\, \dot\partial_l\, X^i,$$

$$b)\quad (\nabla_j \nabla'_k - \nabla'_k \nabla_j)\, X^i = - B_{jkl}^{\cdot\cdot\cdot i}\, X^l,$$

where

$$B_{jkl}^{\cdot\cdot\cdot i} = \Gamma^i_{(j)(k)(l)}.$$

These curvature tensors satisfy the following identity:

$$(1.4)\qquad a)\quad R_{jkl}^{\cdot\cdot\cdot i}\, x'^l = K_{jk}^{\cdot\cdot i},$$

$$b)\quad K_{jk}^{\cdot\cdot i}\, x'^j = H^i_k,$$

$$c)\quad H^i_i = (n-1)\, H.$$

Let us consider the infinitesimal point transformation

$$(1.5)\qquad \bar{x}^i = x^i + v^i(x)\, d\tau.$$

where $d\tau$ is an infinitesimal constant and v^i is the vector defining the infinitesimal transformation which depends only on the positional coordinates. The Lie-derivative of X^i and $\Gamma^i_{(j)(k)}$ [5] are given by

$$(1.6)\qquad \mathscr{L}X^i = \nabla_\gamma\, X^i v^\gamma + \dot\partial_l\, X^i\, \nabla_\gamma\, v^l\, x'^\gamma - X^l\, \nabla_l\, v^i$$

and

$$(1.7)\qquad \mathscr{L}\Gamma^i_{(j)(k)} = \nabla_j\, \nabla_k\, v^i + R_{jkl}^{\cdot\cdot\cdot i}\, v^l + \dot\partial_a\, \Gamma^i_{(j)(k)}\, \nabla_b\, v^a\, x'^b.$$

The following relations hold good in the special Kawaguchi space:

$$(1.8)\qquad a)\quad \mathscr{L}(\dot\partial_l\, T^i_j) - \dot\partial_l(\mathscr{L}T^i_j) = 0,$$

$$b)\quad \nabla_l(\mathscr{L}T^i_j) - \mathscr{L}(\nabla_l\, T^i_j) = \mathscr{L}\Gamma^\gamma_{(j)(l)}\, T^i_\gamma - \mathscr{L}\Gamma^i_{(\gamma)(l)}\, T^\gamma_j + \dot\partial_\gamma\, T^i_j\, \mathscr{L}\Gamma^\gamma_{(h)(m)}\, x'^m$$

and

$$c)\quad \nabla_l(\mathscr{L}\Gamma^i_{(j)(k)}) - \nabla_k(\mathscr{L}\Gamma^i_{(j)(l)}) = \mathscr{L}R_{klj}^{\cdot\cdot\cdot i} + 2\, \dot\partial_a\, \Gamma^i_{(j)[(k)}\, \mathscr{L}\Gamma^a_{(l)](b)}\, x'^b,$$

where

$$(1.9)\qquad R_{jkl}^{\cdot\cdot\cdot i} = 2\,\{\partial_{[k}\, \Gamma^i_{(j)](l)} + \Gamma^h_{(l)[(j)}\, \Gamma^i_{(k)](h)} + \Gamma^h_{[(j)}\, \Gamma^i_{(k)](l)(h)}\}.$$

2. SPECIAL INFINITESIMAL PROJECTIVE TRANSFORMATION

If an infinitesimal transformation (1.5) transforms the system of paths into the same system then it is called an infinitesimal projective transformation of a special Kawaguchi space K_n. With the help of the projective transformation [3], the Lie-derivative of $\Gamma^i_{(j)(k)}$ is given by

$$(2.1)\qquad \mathscr{L}\Gamma^i_{(j)(k)} = \alpha_{(j)}\, \delta^i_k + \alpha_{(k)}\, \delta^i_j + \alpha_{(j)(k)}\, x'^i,$$

where $\alpha(x, x')$ is an arbitrary scalar homogeneous function of degree one with respect to x'^i. If we consider the special projective change given by

$$(2.2)\qquad \alpha = -\frac{1}{n+1}\Gamma^h_{(h)} \quad , \quad \alpha_{(j)} = -\frac{1}{n+1}\Gamma^h_{(h)(j)} ,$$

$$\alpha_{(j)(k)} = -\frac{1}{n+1}\Gamma^h_{(h)(j)(k)} ,$$

then we call the infinitesimal transformation (1.5) special infinitesimal projective transformation in a special Kawaguchi space K_n. In view of (2.2), (2.1) takes the form

$$(2.3)\qquad \mathcal{L}\Gamma^i_{(j)(k)} = \Pi^i_{jk} - \Pi^{*i}_{jk} .$$

The function $\Pi^i_{jk}(x, x')$ is defined by S. Kawaguchi [3] as

$$(2.4)\qquad \Pi^i_{jk} = \Gamma^i_{(j)(k)} - \frac{1}{n+1}\Gamma^h_{(h)(j)(k)} x'^i$$

and

$$(2.5)\qquad \Pi^{*i}_{jk} = \Gamma^i_{(j)(k)} + \frac{1}{n+1}(\Gamma^\gamma_{(\gamma)(j)}\delta^i_k + \Gamma^\gamma_{(\gamma)(k)}\delta^i_j) .$$

In view of the symmetric properties of these functions, the following identities hold:

$$(2.6)\qquad a)\quad \Pi^i_{hi} = \Pi^i_{h(i)} = \Pi^i_{i(h)} = \Gamma^i_{(i)(h)}$$

$$\text{since}\quad \Pi^i_{hk} x'^h = \Pi^i_k = \Gamma^i_{(k)} ,$$

$$b)\quad \Pi^{*i}_{hi} = 2\,\Gamma^i_{(h)(i)} = 2\,\Pi^i_{hi}$$

$$c)\quad \Pi^{*i}_k - \Pi^i_k = \frac{1}{n+1}(\Pi^\gamma_\gamma\,\delta^i_k + \Pi^\gamma_{\gamma k} x'^i) ,$$

where

$$\Pi^{*i}_{hk} x'^h \overset{\text{def}}{=} \Pi^{*i}_k$$

and

$$d)\quad (\Pi^{*i}_k - \Pi^i_k) x'^k = \frac{2}{n+1}\Pi^\gamma_\gamma x'^i .$$

Using relations (2.3), (2.4) and (2.5), we have

$$(2.7)\qquad a)\quad \mathcal{L}B_{ikl}{}^{\cdots i} = \mathcal{L}\Gamma^i_{(i)(k)(l)} = -\Gamma^i_{(i)(k)(l)}$$

$$b)\quad \mathcal{L}B_{iklh}{}^{\cdots\cdot i} = \mathcal{L}\Gamma^i_{(i)(k)(l)(h)} = -\Gamma^i_{(i)(k)(l)(h)} .$$

By virtue of relations (2.3) and (1.8 c), we get

$$(2.8)\qquad \mathcal{L}R_{klj}{}^{\cdots i} = \nabla_l(\Pi^i_{jk} - \Pi^{*i}_{jk}) - \nabla_k(\Pi^i_{jl} - \Pi^{*i}_{jl})$$

$$- 2\,\partial'_a\,\Gamma^i_{(j)[(k)}(\Pi^a_{l]b} - \Pi^{*a}_{l]b})\,x'^b .$$

The tensors P_{kl} and Q_{kl} [3] are given by

$$\text{(2.9)} \qquad a) \quad P_{kl} = \frac{n}{n^2-1} R_{kml}^{\cdot\cdot\cdot m} + \frac{1}{n^2-1} R_{lmk}^{\cdot\cdot\cdot m}.$$

$$b) \quad Q_{kl} = \dot{\partial}_l (P_{kb}\, x'^b),$$

where

$$c) \quad P_{kl}\, x'^l = Q_{kl}\, x'^l.$$

In the paper [3], the following curvature tensors have been obtained:

$$\text{(2.10)} \qquad a) \quad W^i_{jk} = \{ R_{jkl}^{\cdot\cdot\cdot i} + \delta^i_j Q_{kl} - \delta^i_k Q_{jl} - \delta^i_l (Q_{jk} - Q_{kj}) \} x'^l,$$

$$b) \quad W^i_{jkl} = R_{jkl}^{\cdot\cdot\cdot i} + \delta^i_j Q_{kl} - \delta^i_k Q_{jl} - \delta^i_l (Q_{jk} - Q_{kj}) - x'^i \dot{\partial}_l (Q_{jk} - Q_{kj})$$

and

$$c) \quad U^i_{jkl} = B_{jkl}^{\cdot\cdot\cdot i} - \frac{\delta^i_j}{n+1} B_{\alpha kl}^{\cdot\cdot\cdot\alpha} \frac{\delta^i_k}{n+1} B_{\alpha lj}^{\cdot\cdot\cdot\alpha} - \frac{\delta^i_l}{n+1} B_{\alpha jk}^{\cdot\cdot\cdot\alpha} - \frac{x'^i}{n+1} B_{\alpha jkl}^{\cdot\cdot\cdot\cdot\alpha}.$$

Transvecting (2.8) by x'^j and x'^k and using the relations (1.4 *a*), (1.4 *b*), (2.6 *c*) and (2.6 *d*), we get

$$\text{(2.11)} \qquad \mathscr{L} H^i_l = -\frac{2}{n+1} \nabla_l \Pi^\gamma_\gamma x'^i + \frac{1}{n+1} \nabla_k (\Pi^\gamma_\gamma \delta^i_l + \Pi^\gamma_{\gamma l} x'^i) x'^k.$$

Contracting (2.11) with respect to indices i and l and using (1.4 *c*), we have

$$\text{(2.12)} \qquad \mathscr{L} H = \frac{1}{n+1} \nabla_k \Pi^\gamma_\gamma x'^k.$$

In view of (2.11) and (2.12), we can easily obtain

$$\text{(2.13)} \qquad \mathscr{L} W^i_l = 0,$$

where

$$W^i_l \overset{\text{def}}{=} H^i_l - H\delta^i_l - \frac{1}{n+1} (\dot{\delta}_\gamma H^\gamma_l - \dot{\delta}_l H) x'^i.$$

Using relations (2.6 *b*), (2.6 *c*), (2.6 *d*), (2.8), (2.9 *a*) and (2.9 *b*), we get

$$\text{(2.14)} \qquad a) \quad \mathscr{L} P_{kl} = -\frac{1}{n^2-1} \nabla_i \Gamma^\gamma_{(\gamma)(l)(k)} x'^i + \frac{1}{n+1} \nabla_k \Pi^i_{il} + \frac{2}{n^2-1} \Gamma^\gamma_{(\gamma)(l)(k)} \Pi^\gamma_\gamma$$

$$b) \quad \mathscr{L} Q_{kl} = \frac{1}{n+1} \nabla_k \Pi^i_{il},$$

which preserves the identity (2.9 *c*) under the special infinitesimal projective transformation. With the help of three sets of relations (2.10 *a*), (2.14 *b*), (2.8), (2.6); (2.10 *b*), (2.14 *b*), (2.8), (2.6) and (2.7 *a*), (2.7 *b*), (2.10 *c*) the projective curvature tensors have their Lie-derivatives equal to zero. Accordingly, we have.

THEOREM 2.1. *The Lie-derivatives of projective tensors* W^i_j, W^i_{jk}, W^i_{jkl} *and* U^i_{jkl} *vanish automatically under the special infinitesimal projective change.*

Let us suppose that the space is projectively flat (i.e. $W^i_{jkl} = 0$). Then, transvecting the equation (2.10 *b*) by x'^l, we obtain

$$\mathscr{L}K_{jk}^{\cdot\cdot i} = \mathscr{L}\{Q_{jk} - Q_{kj}\}\, x'^i - \delta^i_j\, \mathscr{L}Q_{kl}\, x'^l + \delta^i_k\, \mathscr{L}Q_{jl}\, x'^l. \tag{2.15}$$

Transvecting (2.15) by x'^j and using relations (1.4 *b*) and (1.4 *c*), we obtain the result (2.12). Therefore, we have

THEOREM 2.2. *In a projective flat space the relation* (2.12) *holds good under the special infinitesimal projective transformation.*

Let us suppose that $Q_{jk} = Q_{kj}$, then we have

$$\mathscr{L}(Q_{jk} - Q_{kj}) = 0. \tag{2.16}$$

From equation (2.10 *b*), we find

$$\mathscr{L}R_{jkl}^{\cdot\cdot\cdot i} = -\,\delta^i_j\, \mathscr{L}Q_{kl} + \delta^i_k\, \mathscr{L}Q_{jl}. \tag{2.17}$$

Contracting it with respect to indices i and j we have

$$\mathscr{L}Q_{kl} = -\frac{1}{n+1}\, \mathscr{L}R_{akl}^{\cdot\cdot\cdot a}. \tag{2.18}$$

Using equations (2.16) and (2.18), we obtain

$$\mathscr{L}R_{akl}^{\cdot\cdot\cdot a} = \mathscr{L}R_{alk}^{\cdot\cdot\cdot a}. \tag{2.19}$$

Taking the Lie-derivative of (2.9 *a*) and using (2.19), we get

$$\mathscr{L}Q_{kl} = \mathscr{L}P_{kl}. \tag{2.20}$$

In view of equations (2.14 *a*) and (2.14 *b*) the relation (2.20) holds only when

$$\nabla_i\, \Gamma^{\gamma}_{(\gamma)(l)(k)}\, x'^i = 2\, \Gamma^{\gamma}_{(\gamma)(l)(k)}\, \Pi^{\gamma}_{\gamma} \tag{2.21}$$

is possible. Thus we have

THEOREM 2.3. *If in a special Kawaguchi pace* K_n, $\mathscr{L}(Q_{jk} - Q_{kj}) = 0$, *then the relation* (2.21) *is true under the special infinitesimal projective transformation.*

Using relations (1.8 *b*) and (2.13) for the projective deviation tensor, we obtain

$$\mathscr{L}(\nabla_l\, W^i_j) = (\mathscr{L}\Gamma^i_{(l)(\gamma)})\, W^{\gamma}_j - (\mathscr{L}\Gamma^{\gamma}_{(l)(j)})\, W^i_{\gamma} - (\mathscr{L}\Gamma^{\gamma}_{(l)})\, \dot{\partial}_{\gamma}\, W^i_j. \tag{2.22}$$

Multiplying (2.22) by x'^l and using relations (2.3), (2.6 *c*) and $W^i_j x'^j = 0$, we get

$$\mathscr{L}(\nabla_l W^i_j)\, x'^l = -\frac{1}{n+1}\, \Pi^\alpha_{\alpha\gamma}\, x'^i\, W^\gamma_j + \frac{4}{n+1}\, \Pi^\alpha_\alpha\, W^i_j . \tag{2.23}$$

Contracting (2.22) with respect to indices i and l and using the homogeneity property of W^i_j, we obtain

$$\mathscr{L}(\nabla_i W^i_j) = -\frac{n-2}{n+1}\, \Pi^\alpha_{\alpha\gamma}\, W^\gamma_j . \tag{2.24}$$

Let us now suppose that the special infinitesimal projective transformation leaves invariant the covariant derivative of the projective deviation tensor. Hence for $n > 2$, the equation (2.24) becomes

$$\Pi^\alpha_{\alpha\gamma}\, W^\gamma_j = 0 . \tag{2.25}$$

In view of (2.24) and (2.25), the equation (2.23) becomes

$$\Pi^\alpha_\alpha\, W^i_j = 0 , \tag{.26}$$

which reduces to $W^i_j = 0$. Hence, we have

THEOREM 2.4. *If the special infinitesimal projective transformation leaves invariant the covariant derivative of the projective deviation tensor then the special Kawaguchi space* K_n $(n > 2)$ *will be of constant curvature.*

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