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Decomposition of Berwald’s curvature tensor field in second order recurrent Finsler space

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RIASSUNTO. — Gli Autori studiano spazi di Finsler ricorrenti di secondo ordine: cioè tali che le componenti delle derivate seconde covarianti del tensore di Berwald si esprimano come prodotti delle componenti stesse per quelle di un campo tensoriale doppio.

1. INTRODUCTION

Consider an $n$-dimensional Finsler space $F_n$ with a line element $(x, \dot{x})$ in which the metric is defined by the fundamental function $F(x, \dot{x})$. The function $F(x, \dot{x})$ is positive homogeneous of the first degree in $\dot{x}^i$. The metric tensor is given by

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \delta_i \delta_j F^{2}(x, \dot{x}) .$$

This tensor is evidently positive homogeneous of degree zero in $\dot{x}^i$ and symmetric in the lower indices $i$ and $j$. In the usual manner the contravariant components $g^{ij}$ of the metric tensor is determined by $g^{ik} g_{jk} = \delta^i_j$. The Berwald curvature tensors and their contracted forms are: [3] \(^{(1)}\)

$$H_{jk}^i = 2 \delta_i \delta_j G^i + 2 G^i_{\alpha \beta} \delta j G^\gamma ,$$

$$H_{jk} = 2 \delta_{[i} \delta_{j]} G^i + 2 G^i_{\alpha \beta} \delta_{[i} \delta_{j]} G^\gamma$$

and

$$(1.4) \quad \begin{align*}
(i) & \quad H_{ij} = H_{ij}^\gamma = \delta_{i} \delta_{j} H^\gamma , \\
(ii) & \quad 2 H_{[i,j]} = H_{[i,j]}^\gamma , \\
(iii) & \quad H_{i} = H_{i}^\gamma , \\
(iv) & \quad H = H_{i}^\gamma (n - 1) .
\end{align*}$$

These curvature tensors satisfy the following identities: [3]

$$(1.5) \quad \begin{align*}
(i) & \quad H_{jk}^i = - H_{jk}^i , \\
(ii) & \quad H_{jk}^i \dot{x}^i = 0 , \\
(iii) & \quad H_{jk} \dot{x}^i = H_{ji} , \\
(iv) & \quad \delta_{\gamma} H_{ji}^\gamma \dot{x}^i + (n - 1) H = 0 .
\end{align*}$$

The commutative formulae for Berwald’s curvature tensor fields are

$$(1.6) \quad T_{j}(\theta(\dot{\theta})) - T_{j}(\dot{\theta}(\theta)) = - \delta_{\gamma} T_{j}^i H_{\gamma k}^i + T_{j}^i H_{\gamma k}^i - T_{\gamma}^i H_{j k}^i$$

and

$$(1.7) \quad (\delta_{\dot{x}} T_{j}^i)(\dot{\theta}) - (\delta_{\dot{\theta}} T_{j}^i)(\theta) = T_{\dot{x}}^i G_{j k}^i - T_{\dot{\theta}}^i G_{j k}^i .$$


\(^{(1)}\) The numbers in the brackets refer to the references at the end of the paper. The notation $\partial_i = \partial / \partial x^i$ and $\delta_i = \partial / \partial \dot{x}^i$. 

The notation $\partial_i = \partial / \partial x^i$ and $\delta_i = \partial / \partial \dot{x}^i$. 

\(^{(1)}\) Nella seduta del 14 dicembre 1974.
A recurrent Finsler space is characterised by the property that its Berwald's curvature tensor field $H^i_{jkh}$ satisfies the following relation.

Recurrent Finsler space of the first order:

\[(1.8)\quad H^i_{jkh(i)} = V_i H^i_{jkh}.\]

Recurrent Finsler space of the second order:

\[(1.9)\quad H^i_{jkh(0,m)} = a_{lm} H^i_{jkh}.\]

$V_i$ and $a_{lm}$ are called recurrent vector and recurrence tensor field of first and second order respectively.

2. Decomposition of Berwald's curvature tensor field $H^i_{jkh}$

We consider the decomposition of the Berwald's curvature tensor field $H^i_{jkh}$ in the following manner, [4]

\[(2.1)\quad H^i_{jkh} = X^i_j \varphi_{kh},\]

where $\varphi_{kh}(x, \dot{x})$ and $X^i_j(x, \dot{x})$ are two tensor fields such that

$X^i_j \dot{v}_i = p_j,$

where $p_j$ is non-zero vector field.

The curvature tensor field $H^i_{jk}$ may be put as

\[(2.2)\quad H^i_{jk} = X^i_j \varphi_{jk} \dot{x}^\nu.\]

The main theorems proved in the recent paper [4] are as follows:

**Theorem 2.1.** The recurrent vector field $V_i$ and the tensor field $X^i_j$ behave like a recurrent vector and a recurrent tensor field with respect to the decomposition (2.1), if $V_i$ is independent of $\dot{x}^i$ and they are given by

\[(2.3)\quad V_i(m) = \lambda_m V_i\]

and

\[(2.4)\quad X^i_{j(m)} = \mu_m X^i_j,\]

where $\lambda_m$ and $\mu_m$ are non-zero vector fields.

**Theorem 2.2.** Under the decomposition (2.1) the decomposed vector field $p_j$ and the tensor field $\varphi_{kh}$ behave respectively like a recurrent vector and recurrent tensor field as given by

\[(2.5)\quad p_{j(m)} = (\lambda_m + \mu_m) p_j\]

and

\[(2.6)\quad \varphi_{kh(m)} = (\nu_m - \mu_m) \varphi_{kh}.\]

We prove the following theorems for second order recurrent tensor field in $F_n$. 
THEOREM 2.3. By virtue of the decomposition (2.1), if $V_i$ is independent of direction, the necessary and sufficient condition in order that $\varphi_{kh}$ behaves like a second order recurrent tensor field is that

$$\varphi_{i(m)} = \varphi_{i} \varphi_{m} - V_{m} \varphi_{i} - V_{i} \varphi_{m}.$$  

*Proof.* By successive covariant differentiation of the equation (2.1) with respect to $x^l$ and $x^m$ respectively we have

$$H^i_{jkl}(l) = X^i_{j(l)} \varphi_{kh} + X^i_{j(m)} \varphi_{kh(l)} + X^i_{j} \varphi_{kh(l)m}.$$  

By virtue of (1.9) and using relations (2.1), (2.4) and (2.6) in the equation (2.8), we get

$$a_{lm} \varphi_{kh} = \varphi_{kh(l)m} = \varphi_{kh(l)m} - (\varphi_{i(m)} - \varphi_{i} \varphi_{m} + V_{m} \varphi_{i} + V_{i} \varphi_{m}) \varphi_{kh}.$$  

Now if we suppose that the condition (2.7) is true then

$$\varphi_{kh(l)m} = a_{lm} \varphi_{kh}.$$  

Conversely if (2.10) holds true, from the equation (2.9) we have

$$a_{lm} \varphi_{kh} = \varphi_{kh(l)m} - (\varphi_{i(m)} - \varphi_{i} \varphi_{m} + V_{m} \varphi_{i} + V_{i} \varphi_{m}) \varphi_{kh} = 0$$  

which proves the sufficiency part of the theorem.

THEOREM 2.4. The skew symmetric part of the recurrence tensor field satisfies the relation

$$\varphi_{kh(l)m} - \varphi_{kh(m)(l)} = (a_{lm} - a_{ml}) \varphi_{kh} + (\varphi_{ml(l)} - \varphi_{l(m)l}) \varphi_{kh},$$  

where the scalar function

$$\alpha \overset{\text{def}}{=} \delta_{\gamma} X^i_{\gamma} \delta^i - X^i_{\gamma}.$$  

*Proof.* Subtracting the result obtained by interchanging the indices $l$ and $m$ in (2.9) we have

$$a_{lm} \varphi_{kh} = \varphi_{kh(l)m} = \varphi_{kh(l)m} - (\varphi_{i(m)} - \varphi_{i} \varphi_{m} + V_{m} \varphi_{i} + V_{i} \varphi_{m}) \varphi_{kh}.$$  

Applying the commutation formula (1.6) in the above equation and using the relations (1.4 (i), (ii)), (2.1), (2.2) and $\varphi_{kh} = \varphi_{kh}$, we obtain

$$\{a_{lm} - a_{ml} \} \varphi_{kh} = \{\varphi_{ml(l)} - \varphi_{l(m)l} \} \varphi_{kh} = - \delta_{\gamma} \varphi_{kh} X^i_{\gamma} \delta^i + X^i_{\gamma} \varphi_{kh} \varphi_{lm}.$$  

Differentiating (2.14) covariantly with respect to $x^u$ and using equations (2.6) and (2.14) we obtain

$$\{a_{lm} - a_{ml} \} \varphi_{kh} = \{\varphi_{ml(l)} - \varphi_{l(m)l} \} \varphi_{kh} = \{ - \delta_{\gamma} \varphi_{kh} X^i_{\gamma} \delta^i + X^i_{\gamma} \varphi_{kh} \varphi_{lm} \} \varphi_{lm}.$$
Again differentiating (2.2) with respect to \( x^i \) and using the decomposition of \( H_{jhk} \), we have

\[
- \delta^j_i \varphi_{hk} X^j_i = \delta^j_i X^j_i \varphi_{hk} X^i_i.
\]

By virtue of (2.12 b), (2.15) and (2.16), we obtain the Theorem 2.4.

**Theorem 2.5.** Under the decomposition (2.1) the recurrence tensor field \( a_{lm} \) and the decomposed tensor \( \varphi_{lm} \) satisfy the following relation:

\[
(2.17 \ a) \quad (a_{[lm]} + [\mu_{(lm)}]) + (a_{[im]} + [\mu_{il(m)}]) V_{ln} + (a_{[ln]} + [\mu_{il(n)}]) V_{mn} = \frac{1}{6} (\beta^l_{lm} \varphi_{mn} + \beta^m_{ml} \varphi_{nl} + \beta^i_{il} \varphi_{lm}),
\]

where

\[
(2.17 \ b) \quad \beta^l = \delta^l_i \mu_i X^j_i = \mu_i (X^j_i - \delta^j_i X^i_i x^i).
\]

and \( V_i \) is independent of direction.

**Proof.** Differentiating (2.9) covariantly with respect to \( x^m \) we get

\[
(2.18) \quad \dot{a}^{lm(n)} = a_{lm(n)} \varphi_{kh} + a_{ln} \varphi_{hk(m)} + J_{lm(n)} \varphi_{kh} + J_{ln} \varphi_{hk(n)};
\]

where we have put

\[
J_{lm} = \mu_{lm} - \mu_l \mu_m + V_{mn} \mu_l + V_l \mu_m.
\]

Subtracting the result obtained by interchanging the indices \( m \) and \( n \) in (2.18) and then applying the commutation formula (1.6) we get

\[
(2.19) \quad (a_{lm(n)} - a_{ln(m)}) \varphi_{kh} + (a_{lm} \varphi_{hk(n)} - a_{ln} \varphi_{kh(m)}) + (J_{lm(n)} - J_{ln(m)}) \varphi_{kh} +
+ J_{lm} \varphi_{kh(n)} - J_{ln} \varphi_{kh(m)} = - \delta^j_i \varphi_{kh(l)} H^i_{mn} - \varphi_{t(kh(l)} H^i_{mhn} - \varphi_{t(kh(m)} H^i_{hn}.
\]

With the help of equations (1.4 (i), (ii)), (2.1), (2.2), (2.5) and (2.6), the equation (2.19) takes the form

\[
(2.20) \quad \{ (a_{lm(n)} - a_{ln(m)}) + (a_{lm} V_{mn} - a_{ln} V_{mn}) + (a_{ln} \mu_m - a_{lm} \mu_n) +
+ (J_{lm(n)} - J_{ln(m)}) + J_{lm} (V_{mn} - \mu_m) - J_{ln} (V_{mn} - \mu_n) \} \varphi_{kh} = \{ (\delta^j_i \mu_i) \varphi_{kh} -
- (V_i - \mu_i) \delta_i \varphi_{kh} \} Y^i_{l(m)} \varphi_{mn} - \{ (V_i - \mu_i) X^i_{l(m)} + (V_i - \mu_i) X^i_{l(n)} \} \varphi_{kh} \varphi_{lm},
\]

where \( V_i \) is taken as independent of direction.

Adding the expressions obtained by making a cyclic rotation in the indices \( l, m \) and \( n \) of the equation (2.20) and using the following results: [3]

\[
\varphi_{kh} + \varphi_{kjh} + \varphi_{jk} \varphi_{kh} = 0, \quad V_{j} \varphi_{kh} + V_{k} \varphi_{jh} + V_{h} \varphi_{jk} = 0
\]
and the equation (2.16) we get,

\[ 6 \{ (a_{lm(n)} + \mu_{l(m)n}) + (a_{lm} V_n + \mu_{l(m)} V_m) + (a_{ln} V_m + \mu_{l(n)} V_m) \} \varphi_{kk} = \]

\[ = (\beta_l \varphi_{mn} + \beta_m \varphi_{nl} + \beta_n \varphi_{lm}) \varphi_{kk}, \]

where \( \beta_l \) is given by the equation (2.17 b).

The equation (2.21) completes the proof of the Theorem 2.5.

**Theorem 2.6.** If \( V_i \) is independent of direction, the tensor field \( \varphi_{kk} \) satisfy the relation

\[ (2.22 \ a) \quad 6 \varphi_{kk(l)(m)(n)} = (\beta_l \varphi_{mn} + \beta_m \varphi_{nl} + \beta_n \varphi_{lm}) \varphi_{kk}, \]

where

\[ (2.22 \ b) \quad \beta_l = \delta_{ij} \mu_{j} X^i_t \delta^t_i + \mu_l (X^i_t \delta^{t}_i - \delta_{ij} X^j_t \delta^{t}_j). \]

**Proof.** Subtracting the expressions obtained by a cyclic permutation in the indices \( m \) and \( n \) in (2.18) we obtain

\[ (2.23) \quad \varphi_{kk(l)(m)(n)} - \varphi_{kk(l)(n)(m)} = \{ (a_{lm(n)} - a_{ln(m)}) + (a_{lm} V_n - a_{ln} V_m) + \\
+ (a_{ln} V_m - a_{lm} V_m) + (J_{lm(n)} - J_{ln(m)}) + J_{lm} (V_n - V_m) - J_{ln} (V_m - V_m) \} \varphi_{kk}. \]

By the addition of the expressions obtained by commutating the indices \( l, m \) and \( n \) in (2.23) and using (2.18) we have

\[ (2.24) \quad \varphi_{kk(l)(m)(n)} = \{ (a_{lm(n)} + \mu_{l(m)n}) + (a_{lm} + \mu_{l(m)} V_n) + \\
+ (a_{ln} + \mu_{l(n)} V_m) \} \varphi_{kk}. \]

In view of the Theorem 2.5, the equation (2.24) yields the Theorem 2.6.

**References**