Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Decomposition of Berwald's curvature tensor field in second order recurrent Finsler space

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.6, p. 565–569.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_6_565_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Geometria differenziale. — *Decomposition of Berwald's curvature tensor field in second order recurrent Finsler space.* Nota di H. D. PANDE e T. A. KHAN, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Gli Autori studiano spazi di Finsler ricorrenti di secondo ordine: cioè tali che le componenti delle derivate seconde covarianti del tensore di Berwald si esprimano come prodotti delle componenti stesse per quelle di un campo tensoriale doppio.

1. INTRODUCTION

Consider an n-dimensional Finsler space F_n with a line element $(x, \dot{x})$ in which the metric is defined by the fundamental function $F(x, \dot{x})$. The function $F(x, \dot{x})$ is positive homogeneous of the first degree in $\dot{x}^i$. The metric tensor is given by

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}). \tag{1.1}$$

This tensor is evidently positive homogeneous of degree zero in $\dot{x}^i$ and symmetric in the lower indices i and j. In the usual manner the contravariant components g^{ij} of the metric tensor is determined by $g^{ik} g_{jk} = \delta^i_j$. The Berward curvature tensors and their contracted forms are: [3] (1)

$$H^i_{jk} = 2\, \partial_{[k} \dot{\partial}_{j]} G^i + 2\, G^i_{\gamma[k} \dot{\partial}_{j]} G^\gamma, \tag{1.2}$$

$$H^i_{jkh} = 2\, \partial_{[h} G^i_{k]j} + 2\, G^\gamma_{j[h} G^i_{k]\gamma} + 2\, G^i_{\gamma j[h} \dot{\partial}_{k]} G^\gamma \tag{1.3}$$

and

$$\text{(i)}\quad H_{ij} = H^\gamma_{ij\gamma} = \dot{\partial}_i H_j, \qquad \text{(ii)}\quad 2\, H_{[jk]} = H^\gamma_{\gamma kj}, \tag{1.4}$$

$$\text{(iii)}\quad H_j = H^\gamma_{j\gamma}, \qquad \text{(iv)}\quad H = H^i_i/(n-1).$$

These curvature tensors satisfy the following identities: [3]

$$\text{(i)}\quad H^i_{jkh} = -H^i_{jhk}, \qquad \text{(ii)}\quad H^i_k \dot{x}^k = 0, \tag{1.5}$$

$$\text{(iii)}\quad H_{jk} \dot{x}^j = H_k, \qquad \text{(iv)}\quad \dot{\partial}_\gamma H^\gamma_j \dot{x}^j + (n-1) H = 0.$$

The commutative formulae for Berwald's curvature tensor fields are

$$T^i_{j(h)(k)} - T^i_{j(k)(h)} = -\dot{\partial}_\gamma T^i_j H^\gamma_{hk} + T^\gamma_j H^i_{\gamma hk} - T^i_\gamma H^\gamma_{jhk} \tag{1.6}$$

and

$$(\dot{\partial}_k T^i_j)_{(h)} - \dot{\partial}_k T^i_{j(h)} = T^i_\gamma G^\gamma_{jkh} - T^\gamma_j G^i_{\gamma kh}. \tag{1.7}$$

(*) Nella seduta del 14 dicembre 1974.

(1) The numbers in the brackets refer to the references at the end of the paper. The notation $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

A recurrent Finsler space is characterised by the property that its Berwald's curvature tensor field H^i_{jkh} satisfies the following relation.

Recurrent Finsler space of the first order:

$$H^i_{jkh(l)} = V_l \, H^i_{jkh} . \tag{1.8}$$

Recurrent Finsler space of the second order:

$$H^i_{jkh(l)(m)} = a_{lm} \, H^i_{jkh} . \tag{1.9}$$

V_l and a_{lm} are called recurrent vector and recurrence tensor field of first and second order respectively.

2. Decomposition of Berwald's curvature tensor field H^i_{jkh}

We consider the decomposition of the Berwald's curvature tensor field H^i_{jkh} in the following manner, [4]

$$H^i_{jkh} = X^i_j \, \varphi_{kh} , \tag{2.1}$$

where $\varphi_{kh}(x, \dot{x})$ and $X^i_j(x, \dot{x})$ are two tensor fields such that

$$X^i_j \, v_i = p_j ,$$

where p_j is non-zero vector field.

The curvature tensor field H^i_{jk} may be put as

$$H^i_{jk} = X^i_\gamma \, \varphi_{jk} \, \dot{x}^\gamma . \tag{2.2}$$

The main theorems proved in the recent paper [4] are as follows:

Theorem 2.1. *The recurrent vector field V_i and the tensor field X^i_j behave like a recurrent vector and a recurrent tensor field with respect to the decomposition (2.1), if V_i is independent of $\dot{x}^i$ and they are given by*

$$V_{i(m)} = \lambda_m \, V_i \tag{2.3}$$

and

$$X^i_{j(m)} = \mu_m \, X^i_j , \tag{2.4}$$

where λ_m and μ_m are non-zero vector fields.

Theorem 2.2. *Under the decomposition (2.1) the decomposed vector field p_j and the tensor field φ_{kh} behave respectively like a recurrent vector and recurrent tensor field as given by*

$$p_{j(m)} = (\lambda_m + \mu_m) \, p_j \tag{2.5}$$

and

$$\varphi_{kh(m)} = (v_m - \mu_m) \, \varphi_{kh} . \tag{2.6}$$

We prove the following theorems for second order recurrent tensor field in F_n.

THEOREM 2.3. *By virtue of the decomposition (2.1), if* V_i *is independent of direction, the necessary and sufficient condition in order that* φ_{kh} *behaves like a second order recurrent tensor field is that*

$$\mu_{l(m)} = \mu_l \mu_m - V_m \mu_l - V_l \mu_m . \tag{2.7}$$

Proof. By successive covariant differentiation of the equation (2.1) with respect to x^l and x^m respectively we have

$$H^i_{jkh(l)(m)} = X^i_{j(l)(m)} \varphi_{kh} + X^i_{j(l)} \varphi_{kh(m)} + X^i_{j(m)} \varphi_{kh(l)} + X^i_j \varphi_{kh(l)(m)} . \tag{2.8}$$

By virtue of (1.9) and using relations (2.1), (2.4) and (2.6) in the equation (2.8), we get

$$a_{lm} \varphi_{kh} = \varphi_{kh(l)(m)} + (\mu_{l(m)} - \mu_l \mu_m + V_m \mu_l + V_l \mu_m) \varphi_{kh} . \tag{2.9}$$

Now if we suppose that the condition (2.7) is true then

$$\varphi_{kh(l)(m)} = a_{lm} \varphi_{kh} . \tag{2.10}$$

Conversely if (2.10) holds true, from the equation (2.9) we have

$$(\mu_{l(m)} - \mu_l \mu_m + V_m \mu_l + V_l \mu_m) \varphi_{kh} = 0 \tag{2.11}$$

which proves the sufficiency part of the theorem.

THEOREM 2.4. *The skew symmetric part of the recurrence tensor field satisfies the relation*

$$(a_{lm} - a_{ml})_{(n)} = (\mu_{l(m)} - \mu_{m(l)})_{(n)} + (\alpha_{(n)} + \alpha V_n) \varphi_{lm} - \alpha \mu_n \varphi_{lm} , \tag{2.12 a}$$

where the scalar function

$$\alpha \overset{\text{def}}{=} \dot{\partial}_\gamma X^\gamma_t \dot{x}^t - X^\gamma_\gamma . \tag{2.12 b}$$

Proof. Subtracting the result obtained by interchanging the indices l and m in (2.9) we have

$$\varphi_{kh(l)(m)} - \varphi_{kh(m)(l)} = (a_{lm} - a_{ml}) \varphi_{kh} + (\mu_{m(l)} - \mu_{l(m)}) \varphi_{kh} . \tag{2.13}$$

Applying the commutation formula (1.6) in the above equation and using the relations (1.4 (i), (ii)), (2.1), (2.2) and $\varphi_{kh} = -\varphi_{hk}$, we obtain

$$\{(a_{lm} - a_{ml}) + (\mu_{m(l)} - \mu_{l(m)})\} \varphi_{kh} = (-\dot{\partial}_\gamma \varphi_{kh} X^\gamma_t \dot{x}^t + X^\gamma_\gamma \varphi_{hk}) \varphi_{lm} . \tag{2.14}$$

Differentiating (2.14) covariantly with respect to x^n and using equations (2.6) and (2.14) we obtain

$$\{(a_{lm} - a_{ml})_{(n)} + (\mu_{m(l)} - \mu_{l(m)})_{(n)}\} \varphi_{kh} = (-\dot{\partial}_\gamma \varphi_{kh} X^\gamma_t \dot{x}^t + X^\gamma_\gamma \varphi_{hk})_{(n)} \varphi_{lm} . \tag{2.15}$$

Again differentiating (2.2) with respect to $\dot{x}^i$ and using the decomposition of H^i_{jkh}, we have

$$-\dot{\partial}_\gamma \varphi_{kh} X^\gamma_t \dot{x}^t = \dot{\partial}_\gamma X^\gamma_t \varphi_{kh} \dot{x}^t . \tag{2.16}$$

By virtue of (2.12 *b*), (2.15) and (2.16), we obtain the Theorem 2.4.

THEOREM 2.5. *Under the decomposition (2.1) the recurrence tensor field* a_{lm} *and the decomposed tensor* φ_{lm} *satisfy the following relation:*

$$(a_{[lm(n)]} + \mu_{[l(m)(n)]}) + (a_{[lm} + \mu_{[l(m)}) V_{n]} + (a_{[ln} + \mu_{[l(n)}) \mu_{m]} = \frac{1}{6} (\beta_l \varphi_{mn} + \beta_m \varphi_{nl} + \beta_n \varphi_{lm}) , \tag{2.17 a}$$

where

$$\beta_l = \dot{\partial}_\gamma \mu_l X^\gamma_t \dot{x}^t + \mu_l (X^\gamma_\gamma - \dot{\partial}_\gamma X^\gamma_t \dot{x}^t) \tag{2.17 b}$$

and V_i *is independent of direction.*

Proof. Differentiating (2.9) covariantly with respect to x^n we get

$$\varphi_{kh(l)(m)(n)} = a_{lm(n)} \varphi_{kh} + a_{lm} \varphi_{kh(n)} + J_{lm(n)} \varphi_{kh} + J_{lm} \varphi_{kh(n)} , \tag{2.18}$$

where we have put

$$J_{lm} = \mu_{l(m)} - \mu_l \mu_m + V_m \mu_l + V_l \mu_m .$$

Subtracting the result obtained by interchanging the indices m and n in (2.18) and then applying the commutation formula (1.6) we get

$$(a_{lm(n)} - a_{ln(m)}) \varphi_{kh} + (a_{lm} \varphi_{kh(n)} - a_{ln} \varphi_{kh(m)}) + (J_{lm(n)} - J_{ln(m)}) \varphi_{kh} + \tag{2.19}$$
$$+ J_{lm} \varphi_{kh(n)} - J_{ln} \varphi_{kh(m)} = - \dot{\partial}_\gamma \varphi_{kh(l)} H^\gamma_{mn} - \varphi_{\gamma h(l)} H^\gamma_{kmn} - \varphi_{k\gamma(l)} H^\gamma_{hmn} - \varphi_{kh(\gamma)} H^\gamma_{lmn}.$$

With the help of equations (1.4 (i), (ii)), (2.1), (2.2), (2.5) and (2.6), the equation (2.19) takes the form

$$\{(a_{lm(n)} - a_{ln(m)}) + (a_{lm} V_n - a_{ln} V_m) + (a_{ln} \mu_m - a_{lm} \mu_n) + \tag{2.20}$$
$$+ (J_{lm(n)} - J_{ln(m)}) + J_{lm} (V_n - \mu_n) - J_{ln} (V_m - \mu_m)\} \varphi_{kh} = \{(\dot{\partial}_\gamma \mu_l) \varphi_{kh} -$$
$$- (V_l - \mu_l) \dot{\partial}_\gamma \varphi_{kh}\} X^\gamma_t \dot{x}^t \varphi_{mn} - \{(V_l - \mu_l) X^\gamma_\gamma + (V_\gamma - \mu_\gamma) X^\gamma_l\} \varphi_{kh} \varphi_{lm} ,$$

where V_i is taken as independent of direction.

Adding the expressions obtained by making a cyclic rotation in the indices l, m and n of the equation (2.20) and using the following results: [3]

$$p_j \varphi_{kh} + p_k \varphi_{hj} + p_h \varphi_{jk} = 0 \quad , \quad V_j \varphi_{kh} + V_k \varphi_{hj} + V_h \varphi_{jk} = 0$$

and the equation (2.16) we get,

$$(2.21)\quad 6\{(a_{[lm(n)]}+\mu_{[l(m)(n)]})+(a_{[lm}V_{n]}+\mu_{[lm}V_{n]})+(a_{[ln}\mu_{m]}+\mu_{[l(n)}\mu_{m]})\}\varphi_{kh} = \\ =(\beta_l\varphi_{mn}+\beta_m\varphi_{nl}+\beta_n\varphi_{lm})\varphi_{kh},$$

where β_l is given by the equation (2.17 *b*).

The equation (2.21) completes the proof of the Theorem 2.5.

THEOREM 2.6. *If* V_i *is independent of direction, the tensor field* φ_{kh} *satisfy the relation*

$$(2.22\text{ a})\qquad 6\varphi_{kh[(l)(m)(n)]}=(\beta_l\varphi_{mn}+\beta_m\varphi_{nl}+\beta_n\varphi_{lm})\varphi_{kh}.$$

where

$$(2.22\text{ b})\qquad \beta_l=\dot{\partial}_\gamma\mu_l X_t^\gamma\dot{x}^t+\mu_l(X_\gamma^\gamma-\dot{\partial}_\gamma X_t^\gamma\dot{x}^t).$$

Proof. Subtracting the expressions obtained by a cyclic permutation in the indices *m* and *n* in (2.18) we obtain

$$(2.23)\quad \varphi_{kh(l)(m)(n)}-\varphi_{kh(l)(n)(m)}=\{(a_{lm(n)}-a_{ln(m)})+(a_{lm}V_n-a_{ln}V_m)+ \\ +(a_{ln}\mu_m-a_{lm}\mu_n)+(J_{lm(n)}-J_{ln(m)})+J_{lm}(V_n-\mu_n)-J_{ln}(V_m-\mu_m)\}\varphi_{kh}.$$

By the addition of the expressions obtained by commutating the indices *l*, *m* and *n* in (2.23) and using (2.18) we have

$$(2.24)\qquad \varphi_{kh[(l)(m)(n)]}=\{(a_{[lm(n)]}+\mu_{[l(m)(n)]})+(a_{[lm}+\mu_{[l(m)})V_{n]}+ \\ +(a_{[ln}+\mu_{[l(n)})\mu_{m]}\}\varphi_{kh}.$$

In view of the Theorem 2.5, the equation (2.24) yields the Theorem 2.6.

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