Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Conharmonic transformations of Einstein-Kähler spaces with Bochner curvature tensor

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.6, p. 559–564.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_6_559_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Geometria differenziale. — *Conharmonic transformations of Einstein-Kähler spaces with Bochner curvature tensor.* Nota di S. S. SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — L'Autore, che già si era occupato di spazi Einstein–Kähleriani di tipo speciale, si occupa ora di detti spazi nell'ipotesi che il tensore di curvatura di Bochner soddisfi a speciali relazioni.

1. INTRODUCTION

An $n\,(=2m)$ dimensional Kähler space K^n is a Riemannian space if it admits a structure tensor $\varphi_\mu{}^\lambda$ satisfying

$$\varphi_\mu{}^\alpha \varphi_\alpha{}^\lambda = -\delta_\mu^\lambda \tag{1.1}$$

$$\varphi_{\lambda\mu} = -\varphi_{\mu\lambda} \quad , \quad (\varphi_{\lambda\mu} = \varphi_\lambda{}^\alpha g_{\alpha\mu}) \, , \tag{1.2}$$

and

$$\varphi_\lambda{}^k{}_{,\mu} = 0 \, , \tag{1.3}$$

where the comma followed by an index denotes the operator of covariant differentiation with respect to the metric tensor $g_{\lambda\mu}$ of the Riemannian space.

It is well known that in an Einstein-Kähler space the Bochner curvature tensor reduces to $U_{\lambda\mu\nu}{}^k$ given by

$$U_{\lambda\mu\nu}{}^k = R_{\lambda\mu\nu}{}^k + \frac{R}{n\,(n+2)} [g_{\lambda\nu}\,\delta_\mu^k - g_{\mu\nu}\,\delta_\lambda^k + \varphi_{\lambda\nu}\,\varphi_\mu{}^k - \varphi_{\mu\nu}\,\varphi_\lambda{}^k + 2\,\varphi_{\lambda\mu}\,\varphi_\nu{}^k]. \tag{1.4}$$

A Kähler space is called a space of constant holomorphic sectional curvature if the tensor $U_{\lambda\mu\nu}{}^k$ given by (1.4) vanishes identically.

We shall consider Einstein-Kähler spaces which we shall denote by E^n, satisfying

$$U_{\lambda\mu\nu}{}^k{}_{,\varepsilon} = 0 \tag{1.5}$$

and

$$U_{\lambda\mu\nu}{}^k{}_{,\varepsilon} = K_\varepsilon\, U_{\lambda\mu\nu}{}^k \, , \tag{1.6}$$

where K_ε is a non-zero vector. An Einstein-Kähler space satisfying (1.5) is called a Kähler space with parallel Bochner curvature tensor and that

(*) Nella seduta del 14 dicembre 1974.

(1) Numbers in square brackets refer to the references given at the end.

(2) As for the notations we follow K.B. Lal and S. S. Singh [3] and S. S. Singh [4].

satisfying (1.6) is called a Kähler space with recurrent Bochner curvature tensor. For brevity, such Einstein-Kähler spaces shall be denoted by $Bs(E^n)$ and $Br(E^n)$ respectively.

2. Conharmonic transformation of Einstein–Kähler spaces

Let E^n and $\bar{E}^n$ be two Einstein Kähler spaces. If the metric tensor $\bar{g}_{\lambda\mu}$ of $\bar{E}^n$ is given by

$$\bar{g}_{\lambda\mu} = e^{2\sigma} g_{\lambda\mu} \tag{2.1}$$

where $g_{\lambda\mu}$ is the metric tensor of E^n, then $\bar{E}^n$ is said to be a conformal transformation of E^n.

By a conformal transformation (2.1) the tensor $\varphi_\lambda{}^\mu$ remains unchanged, but the tensor $\varphi_{\lambda\mu}$ is transformed into $\bar{\varphi}_{\lambda\mu}$ by

$$\bar{\varphi}_{\lambda\mu} = e^{2\sigma} \varphi_{\lambda\mu} . \tag{2.2}$$

Also, we have

$$\bar{R}_{\lambda\mu\nu k} = e^{2\sigma} [R_{\lambda\mu\nu k} + g_{\mu\nu} \sigma_{\lambda k} + g_{\lambda k} \sigma_{\mu\nu} - g_{\lambda\nu} \sigma_{\mu k} - g_{\mu k} \sigma_{\lambda\nu}] \tag{2.3}$$

and

$$\bar{R}_{\lambda\mu\nu}{}^k = R_{\lambda\mu\nu}{}^k + g_{\nu\mu} \sigma_\lambda{}^k - g_{\nu\lambda} \sigma_\mu{}^k + \delta_\lambda^k \sigma_{\nu\mu} - \delta_\mu^k \sigma_{\nu\lambda} . \tag{2.4}$$

A conformal transformation with σ satisfying

$$\sigma^\alpha{}_{,\alpha} + \frac{1}{2}(n-2)\, \sigma_\alpha \sigma^\alpha = 0 \tag{2.5}$$

where $\sigma_\alpha = \sigma_{,\alpha}$ and $\sigma^\alpha = g^{\alpha\beta} \sigma_\beta$, has been called a conharmonic transformation and studied by Y. Ishii [1, 2]. A conharmonic transformation satisfies the condition

$$\bar{g}_{\lambda\mu} \bar{R} = g_{\lambda\mu} R \tag{2.6 (3)}$$

which, in view of (1.2) gives

$$\bar{\varphi}_{\lambda\mu} \bar{R} = \varphi_{\lambda\mu} R . \tag{2.7}$$

When an Einstein-Kähler space is transformed into an Einstein-Kähler space by a conharmonic transformation σ satisfies the condition Y. Ishii [1]

$$\sigma_{\mu\nu} = 0 . \tag{2.8}$$

Using (2.8) in (2.4) we have

$$\bar{R}_{\lambda\mu\nu}{}^k = R_{\lambda\mu\nu}{}^k . \tag{2.9}$$

In view of (2.6), (2.7) and (2.9) we obtain

$$\bar{U}_{\lambda\mu\nu}{}^k = U_{\lambda\mu\nu}{}^k . \tag{2.10}$$

(3) Y. Ishii [1].

3. CONHARMONIC TRANSFORMATION OF $Br(E^n)$ SPACES

Differentiating (2.10) covariantly and using the relation

$$\overline{\left\{ {\alpha \atop \lambda\mu} \right\}} = \left\{ {\alpha \atop \lambda\mu} \right\} + \delta_\lambda^\alpha \sigma_\mu + \delta_\mu^\alpha \sigma_\lambda - \sigma^\alpha g_{\lambda\mu} ,$$

we get

$$\bar{U}_{\lambda\mu\nu}{}^k{}_{;\varepsilon} = U_{\lambda\mu\nu}{}^k{}_{,\varepsilon} - 2\,\sigma_\varepsilon U_{\lambda\mu\nu}{}^k - [U_{\lambda\mu\nu\varepsilon}\,\sigma^k + U_{\varepsilon\mu\nu}{}^k \sigma_\lambda + \tag{3.1}$$
$$+ U_{\lambda\varepsilon\nu}{}^k \sigma_\mu + U_{\lambda\mu\varepsilon}{}^k \sigma_\nu] + \sigma^\alpha [U_{\lambda\mu\nu\alpha}\,\delta_\varepsilon^k + U_{\alpha\mu\nu}{}^k g_{\lambda\varepsilon} + U_{\lambda\alpha\nu}{}^k g_{\mu\varepsilon} + U_{\lambda\mu x}{}^k g_{\nu\varepsilon}]$$

where the semi colon denotes the covariant differentiation with respect to $\bar{g}_{\lambda\mu}$.

Now we assume that both E^n and $\bar{E}^n$ are $Br(E^n)$ spaces, so that we have

$$U_{\lambda\mu\nu}{}^k{}_{,\varepsilon} = K_\varepsilon U_{\lambda\mu\nu}{}^k \tag{3.2}$$

$$\bar{U}_{\lambda\mu\nu}{}^k{}_{;\varepsilon} = \bar{K}_\varepsilon \bar{U}_{\lambda\mu\nu}{}^k \tag{3.3}$$

for non-zero vectors K_ε and $\bar{K}_\varepsilon$.

Substituting (3.2) and (3.3) in (3.1) and using (2.10) we obtain

$$(\bar{K}_\varepsilon - K_\varepsilon) U_{\lambda\mu\nu}{}^k = -2\,\sigma_\varepsilon U_{\lambda\mu\nu}{}^k - [U_{\lambda\mu\nu\varepsilon}\,\sigma^k + U_{\varepsilon\mu\nu}{}^k \sigma_\lambda + \tag{3.4}$$
$$+ U_{\lambda\varepsilon\nu}{}^k \sigma_\mu + U_{\lambda\mu\varepsilon}{}^k \sigma_\nu] + \sigma^\alpha [U_{\lambda\mu\nu\alpha}\,\delta_\varepsilon^k + U_{\alpha\mu\nu}{}^k g_{\lambda\varepsilon} + U_{\lambda\alpha\nu}{}^k g_{\mu\varepsilon} + U_{\lambda\mu\alpha}{}^k g_{\nu\varepsilon}].$$

Contracting ε and k in (3.4) and using the relations

$$U_{\alpha\lambda\mu}{}^\alpha = U_{\lambda\alpha\mu}{}^\alpha = U_{\lambda\mu\alpha}{}^\alpha = 0 \tag{3.5}$$

and

$$U_{\lambda\mu\nu}{}^k + U_{\mu\nu\lambda}{}^k + U_{\nu\lambda\mu}{}^k = 0 \tag{3.6}$$

we obtain

$$(\bar{K}_\alpha - K_\alpha) U_{\lambda\mu\nu}{}^\alpha = (n-3)\,\sigma_\alpha U_{\lambda\mu\nu}{}^\alpha . \tag{3.7}$$

Transvecting (3.7) with σ^ν we get

$$(\bar{K}_\alpha - K_\alpha) U_{\lambda\mu\beta}{}^\alpha \sigma^\beta = 0 . \tag{3.8}$$

On the other hand transvection of (3.4) with σ^ε gives

$$(\bar{K}_\alpha - K_\alpha + 2\,\sigma_\alpha) U_{\lambda\mu\nu}{}^k \sigma^\alpha = 0 . \tag{3.9}$$

Hence, we find either

$$U_{\lambda\mu\nu}{}^k = 0 \tag{3.10}$$

or

$$(\bar{K}_\alpha - K_\alpha)\,\sigma^\alpha = -2\,\sigma_\alpha \sigma^\alpha . \tag{3.11}$$

We consider the case in which (3.11) holds. Transvecting (3.4) with $(\bar{K}_k - K_k)\,\sigma^\nu$ we have

$$
\begin{aligned}
(\bar{K}_\varepsilon - K_\varepsilon)(\bar{K}_k - K_k)\,\sigma^\nu\, U_{\lambda\mu\nu}{}^k = {} & -2\,(\bar{K}_k - K_k)\,U_{\lambda\mu\nu}{}^k\,\sigma^\nu\,\sigma_\varepsilon - \\
& - \{(\bar{K}_k - K_k)\,U_{\lambda\mu\nu\varepsilon}\,\sigma^k\,\sigma^\nu + (\bar{K}_k - K_k)\,U_{\varepsilon\mu\nu}{}^k\,\sigma^\nu\,\sigma_\lambda + \\
& + (\bar{K}_k - K_k)\,\sigma^\nu\,\sigma_\mu\,U_{\lambda\varepsilon\nu}{}^k + (\bar{K}_k - K_k)\,U_{\lambda\mu\varepsilon}{}^k\,\sigma_\nu\,\sigma^\nu\} + \\
& + \sigma^\alpha\,\{(\bar{K}_\varepsilon - K_\varepsilon)\,U_{\lambda\mu\nu\alpha}\,\sigma^\nu + (\bar{K}_k - K_k)\,U_{\alpha\mu\nu}{}^k\,g_{\lambda\varepsilon}\,\sigma^\nu + \\
& + (\bar{K}_k - K_k)\,U_{\lambda\alpha\nu}{}^k\,g_{\mu\varepsilon}\,\sigma^\nu + (\bar{K}_k - K_k)\,U_{\lambda\mu\alpha}{}^k\,g_{\nu\varepsilon}\,\sigma^\nu\}
\end{aligned}
$$

which on using equations (3.11), (3.7) and (3.8) reduces to (3.12)

$$(n - 1)\,\sigma_\alpha\,\sigma^\alpha\,U_{\lambda\mu\nu}{}^\beta\,\sigma_\beta = 0\,. \tag{3.12}$$

Hence, we find either

$$\sigma_\alpha\,\sigma^\alpha = 0\,, \tag{3.13}$$

which implies that σ is constant
or

$$\sigma_\beta\,U_{\lambda\mu\nu}{}^\beta = 0\,. \tag{3.14}$$

When (3.14) holds, (3.4) becomes

$$(\bar{K}_\varepsilon - K_\varepsilon)\,U_{\lambda\mu\nu}{}^k = -2\,\sigma_\varepsilon\,U_{\lambda\mu\nu}{}^k - [U_{\lambda\mu\nu\varepsilon}\,\sigma^k + U_{\varepsilon\mu\nu}{}^k\,\sigma_\lambda + U_{\lambda\varepsilon\nu}{}^k\,\sigma_\mu + U_{\lambda\mu\varepsilon}{}^k\,\sigma_\nu].$$

Transvecting this equation with σ_k and using (3.14) we get

$$\sigma_k\,\sigma^k\,U_{\lambda\mu\nu\varepsilon} = 0\,.$$

Hence, we find either (3.10) or (3.13). In the case of (3.13) the equation (3.4) becomes

$$(\bar{K}_\varepsilon - K_\varepsilon)\,U_{\lambda\mu\nu}{}^k = 0\,,$$

from which follows either (3.10) or $\bar{K}_\varepsilon = K_\varepsilon$.

Thus we have

THEOREM 3.1. *If a* $Br(E^n)$ *is transformed into another* $Br(E^n)$ *by a conharmonic transformation, then the following cases occur:*

1) *the space is that of constant holomorphic sectional curvature.*
2) σ *is constant and the recurrence vectors coincide.*

Since an Einstein-Kähler recurrent space is a $Br(E^n)$, from the above theorem we have the following

COROLLARY. *If an Einstein–Kähler space is transformed into another Einstein–Kähler space by a conharmonic transformation as follows:*

a $Br(E^n)$*–space* $\rightarrow$ *a recurrent* E^n*–space*

or

a recurrent E^n*–space* $\rightarrow$ *a* $Br(E^n)$*–space or a recurrent* E^n*–space*

then the space is that of constant holomorphic sectional curvature or σ *is constant and the recurrence vectors coincide.*

Now, if σ is constant and the space is not of constant holomorphic sectional curvature, then (3.1) can be written as

$$\bar{U}_{\lambda\mu\nu}{}^{k}{}_{;\varepsilon} = U_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} .$$

Consequently, a B*r* (E^n) may be transformed into a B*r* (E^n) by a conharmonic transformation.

Therefore, in view of Theorem 3.1, we have

THEOREM 3.2. *The necessary and sufficient condition for a* B*r* (E^n) *which is not of constant holomorphic sectional curvature to be transformed into another* B*r* (E^n) *is that* σ *is constant.*

4. CONHARMONIC TRANSFORMATION OF B*s* (E^n) SPACES

If E^n and $\bar{E}^n$ are both B*s* (E^n), then regarding K_ε and $\bar{K}_\varepsilon$ both zero and proceeding exactly in the same way as in the proof of Theorem 3.1, we find either $U_{\lambda\mu\nu}{}^{k} = 0$ or σ is constant. Hence, we have

THEOREM 4.1. *If a* B*s* (E^n) *is transformed into another* B*s* (E^n) *by a conharmonic transformation, then the space is that of constant holomorphic sectional curvature or* σ *is constant.*

COROLLARY. *If an Einstein-Kähler space is transformed into another Einstein–Kähler space by a conharmonic transformation as follows:*

a B*s* (E^n)*-space* $\rightarrow$ *a symmetric* E^n*-space*

or

a symmetric E^n*-space* $\rightarrow$ *a* B*s* (E^n)*-space or a symmetric* E^n*-space*

then the space is that of constant holomorphic sectional curvature or σ *is constant.*

Also we have

THEOREM 4.2. *A necessary and sufficient condition for a* B*s* (E^n) *which is not of constant holomorphic sectional curvature to be transformed into another* B*s* (E^n) *by a conharmonic transformation is that* σ *is constant.*

Next, we assume that an E^n is a B*r* (E^n) and $\bar{E}^n$ is a B*s* (E^n). Then regarding $\bar{K}_\varepsilon$ as zero and proceeding exactly the same way as in the proof of Theorem 3.1 we find either $U_{\lambda\mu\nu}{}^{k} = 0$ or σ is constant and $K_\varepsilon = \bar{K}_\varepsilon = 0$. However, since K_ε is a non-zero vector, the space must be that of constant holomorphic sectional curvature. Hence, we have

THEOREM 4.3. *If a* B*r* (E^n) *is transformed into a* B*s* (E^n) *or a* B*s* (E^n) *is transformed into a* B*r* (E^n) *by a conharmonic transformation, then the space is that of constant holomorphic sectional curvature.*

Since a Kähler space which is symmetric in the sense of Cartan is a $Bs(E^n)$, from this theorem we have the following.

COROLLARY. *If an* E^n *is transformed into another* E^n *by a conharmonic transformation as follows*:

a $Br(E^n)$*-space* $\rightarrow$ *a symmetric* E^n*-space*
a $Bs(E^n)$*-space* $\rightarrow$ *a recurrent* E^n*-space*
a recurrent E^n*-space* $\rightarrow$ *a* $Bs(E^n)$*-space or a symmetric* E^n*-space*
a symmetric E^n*-space* $\rightarrow$ *a* $Br(E^n)$*-space or a recurrent* E^n*-space*

then, the space must be that of constant holomorphic sectional curvature.

The Author is thankful to prof. K. B. Lal for his helpful guidance.

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