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On the existence of periodic solutions of certain fourth order differential equations

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Equazioni differenziali ordinarie non lineari. — *On the existence of periodic solutions of certain fourth order differential equations.* Nota di H. O. Tẹjumọla, presentata (*) dal Socio G. Sansone.

Riassunto. — L'Autore estendendo un suo precedente Teorema sotto opportune condizioni prova l'esistenza di almeno una soluzione periodica per l'equazione $x^{(4)} + a_1 \dddot{x} + g(\dot{x}) \ddot{x} + a_3 \dot{x} + h(a, \dot{x}, \ddot{x}, \dddot{x}) = p(t)$, con a_1, a_3 costanti, y, h, p funzioni continue dei loro argomenti, $p(t + \omega) = p(t)$.

1. In a recent paper [1] we considered the differential equation.

$$x^{IV} + a_1 \dddot{x} + g(\dot{x}) \ddot{x} + a_3 \dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}) = p(t), \tag{1.1}$$

where $a_1 > 0$, $a_3 > 0$ are constants, g, h and p are continuous functions of the arguments shown in (1.1), $p(t)$ is periodic with period ω and h is bounded, that is

$$|h(x, y, z, u)| \leq H \text{ (H a constant)} \qquad \text{for all } x, y, z, \text{ and } u.$$

We proved the following result by use of the Leray-Schauder fixed point technique:

Theorem 1. *The equation* (1.1) *admits of at least one* ω*-periodic solution if*

(i) $$h(x, y, z, u) \operatorname{sgn} x > 0 \qquad (|x| \geq x_0).$$

(ii) *there are constants* $A_0 \geq 0$, $A_1 \geq 0$ *and* $a_2 > 0$ *satisfying*

$$a_2 > a_1^{-1} a_3 \tag{1.2}$$

such that

$$|P(t)| \equiv \left| \int_0^t p(\tau)\, d\tau \right| \leq A_0 \qquad (t \geq 0),$$

$$G(y) \operatorname{sgn} y \geq a_2 |y| - A_1 \text{ for all } y, \; G(y) \equiv \int_0^y g(\eta)\, d\eta. \tag{1.3}$$

The condition (1.3) together with (1.2) implies that

$$(G(y) - a_1^{-1} a_3 y) \operatorname{sgn} y \geq \gamma |y| - A_1 \qquad \text{for all } y, \gamma = (a_2 - a_1^{-1} a_3),$$

so that

$$(G(y) - a_1^{-1} a_3 y) \operatorname{sgn} y \to +\infty \text{ as } |y| \to \infty. \tag{1.4}$$

(*) Nella seduta del 14 dicembre 1974.

We verify here that the following extension of Theorem 1 holds:

THEOREM 1'. *Subject to the conditions of Theorem 1, but with* (1.4) *in place of* (1.3), *the equation* (1.1) *admits of at least one* ω*-periodic solution.*

The main ingredient for the proof of Theorem 1 in [1] are the results (2.6) and (2.7) of [1 ; § 2]. Our task here is the verification of the corresponding results for Theorem 1' and, from these, Theorem 1' would follow just as in [1].

Consider, as in [1], the parameter μ-dependent equation

$$x^{\mathrm{IV}} + a_1 \dddot{x} + g_\mu(\dot{x})\ddot{x} + a_3\dot{x} + h_\mu(x, \dot{x}, \ddot{x}, \dddot{x}) = \mu p(t), \qquad 0 \leq \mu \leq 1, \tag{1.5}$$

where

$$\begin{cases} g_\mu(\dot{x}) = (1-\mu)a_2 + \mu g(\dot{x}), \\ h_\mu(x, \dot{x}, \ddot{x}, \dddot{x}) = \begin{matrix} (1-\mu)a_4 x + \mu h(x, \dot{x}, \ddot{x}, \dddot{x}), & \text{if } |x| \leq S \\ (1-\mu)a_4 S \operatorname{sgn} x + \mu h(x, \dot{x}, \ddot{x}, \dddot{x}), & \text{if } |x| \geq S. \end{matrix} \end{cases} \tag{1.6}$$

a_2 is a constant satisfying (1.2) and

$$|h_\mu(x, y, z, u) \leq a_4 S + H \qquad \text{for all } x, y, z \text{ and } u.$$

Corresponding to (2.6) and (2.7) of [1], we show here that there is a constant $D_0 > 0$, independent of μ and S, and a continuous function $\nabla(\eta)$ such that every solution $x(t)$ of (1.5) ultimately satisfies

$$|x(t)| \leq x_0 + D_0 \Delta(\eta), \tag{1.7}$$

$$\max(|\dot{x}(t)|, |\ddot{x}(t)|, |\dddot{x}(t)|) \leq \nabla(\eta), \tag{1.8}$$

where $\eta = a_1 a_3^{-1}(a_4 S + H) + A_0 + 1$ is as defined in [1 ; § 2].

2. VERIFICATION OF (1.7) AND (1.8). Adopt all of the notations introduced in [1 ; § 2] and let $V = V_0 - 2\eta V_1 - \eta V_2$ be the function defined by (3.1) and (3.2) of [1]. Corresponding to the estimate (3.3) of [1] we have here that V satisfies

$$\begin{cases} -d_3(\eta^2+1) + \alpha(r) \leq V \leq d_4 r + G^*(r) + d_5(\eta^2+1) & \text{if } |y| \geq d_9 \\ r \equiv y^2 + z^2 + u^2 \end{cases} \tag{2.1}$$

for some d_9, where $\alpha(r)$ is a continuous non-decreasing function satisfying

$$\alpha(r) \to +\infty \qquad \text{as } r \to \infty. \tag{2.2}$$

Indeed, the right hand inequality follows as in [1 ; § 2]. As for the other half, observe from (3.1) and (3.2) of [1] that for some d_{10} V satisfies

$$V \geq V_3 - d_3(\eta^2 + 1). \tag{2.3}$$

where

$$V_3 = \int_0^y (G_\mu(s) - a_1^{-1} a_3 s)\, ds + d_{10} u^2 +$$
$$+ 1/2 (a_1^{1/2} a_3^{-1/2} z - a_1^{-1/2} a_3^{1/2} y)^2,\ G_\mu(y) = \int_0^y g_\mu(s)\, ds .$$

It is easy to verify from (1.6), using (1.4), (1.2) and the continuity of G, that

$$(G_\mu(y) - a_1^{-1} a_3 y) \operatorname{sgn} y \geq 1/2 \min [(a_2 - a_1^{-1} a_3) \mid y \mid ,$$
$$(G(y) - a_1^{-1} a_3 y) \operatorname{sgn} y] \text{ if } \mid y \mid \geq d_{11} ,$$

since $(G(y) - a_1^{-1} a_3 y) \operatorname{sgn} y > 0$ if $\mid y \mid \geq d_{11}$ for d_{11} large enough. Therefore $V_3 \to +\infty$ as $r \to \infty$ and, by the continuity of V_3, there is a continuous non-decreasing function α satisfying (2.2) such that

$$V_3 \geq \alpha(r) \qquad \text{if } \mid y \mid \geq d_{12} ,$$

for some constant d_{12}. This, combined with (2.3), verifies the left hand inequality in (2.1).

As in [1 ; § 3], take (1.5) in the system form

$$\text{(2.4)} \quad \begin{cases} \dot{x} = y ,\ \dot{y} = u - a_1 y ,\ \dot{z} = - a_3 y - h_\mu(x , y , v , w) \\ \dot{u} = z - G_\mu(y) + \mu P(t) ,\ v \equiv u - a_1 y ,\ w \equiv z - G_\mu(y) - \mu P(t) . \end{cases}$$

For precisely the same reasons in [1 ; § 3] $\dot{V}^*$ satisfies

$$\text{(2.5)} \qquad \dot{V}^* \leq -1 \qquad \text{if } r(t) \geq \delta_0^2(\eta)$$

for any solution $(x , y , z , u) \equiv (x(t) , y(t) , z(t) , u(t))$ of (2.4), where $\delta_0(\eta)$ is a continuous functon of η. Following the arguments in [1 ; § 4], the results (2.1) and (2.5) will now be used to obtain (1.8).

Indeed, let $\delta_4^2(\eta) = \delta_0^2(\eta) + d_9^2$. Then, for any solution (x , y , z , u) of (2.4), there is a t_0 such that $r(t_0) < \delta_4^2(\eta)$. For otherwise, $r(t) \geq \delta_4^2(\eta)$ for all t, and by (2.5), $V \to -\infty$ as $t \to \infty$, contrary to (2.1). Now fix $\delta_5(\eta)$ (as is possible, in view of (2.2)) such that

$$\text{(2.6)} \quad \delta_5(\eta) > \delta_4(\eta) \text{ and } \alpha(\delta_5^2(\eta)) \geq (d_3 + d_5) \cdot (\eta^2 + 1) + G^*(\delta_4^2(\eta)) + d_4 \delta_4^2(\eta) .$$

Then

$$\text{(2.7)} \qquad r(t) \leq \delta_5^2(\eta) \qquad \text{for all } t \geq t_0 .$$

For otherwise, there would exist a $T_0 > t_0$ such that $r(T_0) > \delta_5^2(\eta)$ and, by the continuity of $r(t)$, this would imply the exixtence of t_1 , t_2 with $t_2 > t_1 > t_0$ such that

$$\text{(2.8)} \qquad r(t_2) = \delta_5^2(\eta) ,\ r(t_1) = \delta_4^2(\eta)$$

and

$$\text{(2.9)} \qquad r(t) \geq \delta_4^2(\eta) ,\ t_1 \leq t \leq t_2 .$$

But then (2.8), (2.1) and (2.6) show that

$$\begin{aligned} V(t_2) &\geq \alpha(r(t_2)) - d_3(\eta^2 + 1) \\ &\geq d_4\,\delta_4^2(\eta) + G^*(\delta_4^2(\eta)) + d_5(\eta^2 + 1) \\ &\geq V(t_1)\,, \end{aligned}$$

while (2.9) and (2.5) (since $\delta_4^2(\eta) > \delta_0^2(\eta)$) imply that $V(t_2) < V(t_1)$, which is impossible. This verifies (2.7) and the result (1.8) therefore follows. The other result (1.7) can be obtained as in the proof of the corresponding result (2.6) of [1].

All of the arguments in [1 ; § 5] apply here and we conclude as in [1] that there is a choice of a_4 and S such that (1.5) admits of an ω-periodic solution $x(t)$ satisfying $|x(t)| \leq S$ for all t.

The considerations in [1 ; § 6] regarding the equation

$$x^{IV} + f(\ddot{x})\,\dddot{x} + g(\dot{x})\,\ddot{x} + a_3\,\dot{x} + h(x\,,\dot{x}\,,\ddot{x}\,,\dddot{x}) = p(t)$$

also apply here under the same condition ((6.2) of [1]) on f and with g satisfying condition (1.4).

REFERENCE

H. O. TẸJUMỌLA (1974) – *Periodic Solutions of certain fourth order differential equations*, « Rend. Accad. Naz. Lincei », (8), *17*, 328–336.