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A sufficient condition for an exponential dichotomy

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Equazioni differenziali ordinarie. — *A sufficient condition for an exponential dichotomy.* Nota di DAVID LOWELL LOVELADY, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà condizioni sufficienti su $A(t)$ perché l'equazione $u'(t) = f(t) + A(t)u(t)$ abbia soluzioni limitate.

I. INTRODUCTION AND RESULTS

Let Y be a finitedimensional linear space with norm $|\ \ |$, and let $R^+ = [0, \infty)$. Let $\mathcal{A}$ be the algebra of linear functions from Y to Y, with induced norm $\|\ \ \|$, let I be the identity in $\mathcal{A}$, and let A be a locally integrable function from R^+ to $\mathcal{A}$, We propose to study the differential equations

(1) $$v'(t) = A(t)\,v(t)$$

and

(2) $$u'(t) = f(t) + A(t)\,u(t)$$

on R^+, where f is always at least locally integrable.

Let Φ be the fundamental solution of (1), i.e., Φ is that locally absolutely continuous function from R^+ to $\mathcal{A}$ such that

$$\Phi(t) = I + \int_0^t A(s)\,\Phi(s)\,ds$$

whenever t is in R^+, and recall that each value of Φ is invertible. Let M_1 be the subspace of Y to which x belongs if and only if the function from R^+ to Y described by $t \to \Phi(t)\,x$ is bounded. Let M_2 be a subspace of Y such that $Y = M_1 \oplus M_2$, and let P_1 and P_2 be supplementary projections in $\mathcal{A}$ with ranges M_1 and M_2 respectively. Finally, let L from $R^+ \times R^+$ to $\mathcal{A}$ be given by $L(t, s) = \Phi(t)\,P_1\,\Phi(s)^{-1}$ if $0 \leq s \leq t$ and $L(t, s) = -\Phi(t)\,P_2\,\Phi(s)^{-1}$ if $s > t$.

We shall say that $\mathcal{A}$ admits an *exponential dichotomy* if and only if there are positive numbers J and α such that $\|L(t, s)\| \leq J\,e^{-\alpha|t-s|}$ whenever (t, s) is in $R^+ \times R^+$. In [1], T. F. Bridgland, Jr. has shown that if there are positive numbers K and M such that

(3) $$\|L(t, s)\| \leq K$$

(*) Nella seduta del 14 dicembre 1974.

whenever (t, s) is in $R^+ \times R^+$ and

$$\int_0^\infty \| L(t, s) \| ds \leq M \tag{4}$$

whenever t is in R^+, then A admits an exponential dichotomy. W. A. Coppel [3, Theorem 3, p. 134] has shown that (4) and the boundedness of A yield the same conclusion. The main effort of the present work is to extend Bridgland's result to the following theorem.

THEOREM. *Suppose that if c is a positive number then there is a positive number* K_c *such that* $\| L(t, s) \| \leq K_c$ *whenever* (t, s) *is in* $R^+ \times R^+$ *and* $| t - s | \leq c$. *Suppose also that there exists a number q of* $[1, \infty)$ *such that*

$$\sup_{t \geq 0} \int_0^t \| L(t, s) \|^q ds < \infty.$$

Then A *admits an exponential dichotomy.*

Bridgland's hypotheses are known [3, p. 131] to be equivalent to requiring that if f is in either $L^1[R^+, Y]$ or $L^\infty[R^+, Y]$ then (2) has bounded solution. Our present hypotheses have similar interpretations. If $q = 1$, our second hypothesis is the same as Bridgland's. If $q > 1$, and $p = q/(q - 1)$, then R. Conti [2] has shown that our second condition is equivalent to requiring that if f is in $L^p[(R^+, Y]$ then (2) has a bounded solution. Techniques similar to those of Bridgland, Conti, and Coppel can be used to show that our first condition is equivalent to the following: If c is a positive number and F is a bounded subset of $L^1[R^+, Y]$, each member of which has its support in an interval of length c, then there is a bounded subset G of $L^\infty[R^+, Y]$ such that if f is in F there is a solution u of (2) in G.

II. PROOFS

First we shall show that it suffices to consider the case $q = 1$, and then we shall prove the theorem in this case.

Let ω be a positive number such that $\| L(t, s) \| \leq \omega$ whenever (t, s) is in $R^+ \times R^+$ and $| t - s | \leq 1$. Suppose $q > 1$, and let Γ be a positive number such that

$$\int_0^t \| L(t, s) \|^q ds \leq \Gamma^q$$

whenever t is in R^+. Let $\gamma = 1/\Gamma$, and let $p = q/(q - 1)$.

We now claim that if (t, s) is in $R^+ \times R^+$ and $t - s \geq 1$ then

$$\| L(t, s) \| \leq \omega\Gamma (t - s)^{1/p} \exp [\gamma q - \gamma q (t - s)^{1/q}]. \tag{5}$$

If $P_1 = 0$ this is obvious, so assume $P_1 \neq 0$. Let s be in R^+, and let φ be given on $[s, \infty)$ by $\varphi(t) = | L(t, s) |^{-1}$. Now, if t is in $[s, \infty)$,

$$\varphi(t)^{-1} \int_s^t \varphi(r)\, dr = \left\| \left(\int_s^t \varphi(r)\, dr \right) L(t, s) \right\|$$

$$\leq \int_s^t \varphi(r) \, \| L(t, r)\, L(r \cdot s) \| \, dr$$

$$\leq \int_s^t \| L(t, r) \| \, dr$$

$$\leq \left(\int_s^t \| L(t, r) \|^q \, ds \right)^{1/q} (t - s)^{1/p}$$

$$\leq \Gamma (t - s)^{1/p}.$$

For convenience, put $\sigma = s + 1$. Now, if $t \geq \sigma$,

$$\Gamma (t - s)^{1/p} \varphi(t) \geq \int_s^t \varphi(r)\, dr$$

$$= \int_s^\sigma \varphi(r)\, dr + \int_\sigma^t \varphi(r)\, dr$$

$$\geq 1/\omega + \int_\sigma^t \varphi(r)\, dr,$$

so

$$\varphi(t) \geq (\gamma/\omega)\, (t - s)^{-1/p} + \gamma (t - s)^{-1/p} \int_\sigma^t \varphi(r)\, dr.$$

Thus, if $t \geq \sigma$, $\varphi(t) \geq \psi(t)$, where ψ solves

$$\psi(t) = (\gamma/\omega)\, (t - s)^{-1/p} + \gamma (t - s)^{-1/p} \int_\sigma^t \psi(r)\, dr$$

on $[\sigma, \infty)$. Bur ψ is given by

$$\psi(t) = (\gamma/\omega)\, (t - s)^{-1/p} \exp \left[- \gamma q + \gamma q (t - s)^{-1/q} \right],$$

so

$$\| L(t, s) \| = \varphi(t)^{-1} \leq \psi(t)^{-1}$$

yields (5), and our claim is verified.

Next we claim that if $s - t \geq 1$ then

$$\| L(t, s) \| \leq \psi \Gamma (s-t)^{1/p} \exp [(\gamma q - \gamma q (s-t)^{1/q}] . \tag{6}$$

If $P_2 = 0$, this is obvious, so assume $P_2 \neq 0$. Let s be in R^+, and let φ be given on $[0, s]$ by $\varphi(t) = \| L(t, s) \|^{-1}$. As before,

$$\varphi(t)^{-1} \int_t^\sigma \varphi(r) \, dr \leq \Gamma (s-t)^{1/p}$$

if t is in $[0, s]$, so if $\sigma = s - 1$,

$$\varphi(t) \geq (\gamma/\omega)(s-t)^{-1/p} + \gamma (s-t)^{-1/p} \int_t^\sigma \varphi(r) \, dr$$

whenever t is in $[0, \sigma)$. This last integral inequality is somewhat unorthodox, so we shall give more detail to solving it. Let β be given on $[0, \sigma]$ by

$$\beta(t) = \int_t^\sigma \varphi(r) \, dr .$$

Now $\beta(\sigma) = 0$, and if t is in $[0, \sigma)$,

$$- \beta'(t) \geq (\gamma/\omega)(s-t)^{-1/p} + \gamma (s-t)^{-1/p} \beta(t) ,$$

$$\beta'(t) \leq - (\gamma/\omega)(s-t)^{-1/p} - \gamma (s-t)^{-1/p} \beta(t) ,$$

$$\beta'(t) + \gamma (s-t)^{-1/p} \beta(t) \leq - (\gamma/\omega)(s-t)^{-1/p} ,$$

$$(\beta(t) \exp [- \gamma q (s-t)^{1/q}])' \leq - (\gamma/\omega)(s-t)^{-1/p} \exp (- \gamma q (s-t)^{1/q}] .$$

Integrating this last inequality yields

$$- \beta(t) \exp [- \gamma q (s-t)^{1/q}] \leq - (\gamma/\omega) \int_t^\sigma (s-r)^{-1/p} \exp (\gamma q (s-r)^{1/q}] \, dr$$

$$= - (1/\omega) e^{-\gamma q} + (1/\omega) \exp [- \gamma q (s-t)^{1/q}] ,$$

$$\beta(t) \geq (1/\omega) \exp (\gamma q + \gamma q (s-t)^{1/q}] - (1/\omega) .$$

Thus, if t is in $[0, \sigma)$,

$$\varphi(t) \geq (\gamma/\omega)(s-t)^{-1/p} + \gamma (s-t)^{-1/p} \beta(t)$$

$$\geq (\gamma/\omega)(s-t)^{-1/p} \exp [- \gamma q + \gamma q (s-t)^{1/q}] ,$$

(6) follows, and our second claim is verified.

Suppose that $t \geq 1$. From (5) we have

$$\int_0^{t-1} \| L(t, s) \| \, ds \leq \omega \Gamma e^{\gamma q} \int_0^{t-1} (t-s)^{1/p} \exp[-\gamma q (t-s)^{1/q}] \, ds$$

$$= (\omega \Gamma^2 / q) \, e^{\gamma q} (e^{-\gamma q} - \exp(-\gamma q t^{1/q}))$$

$$\leq \omega \Gamma^2 / q .$$

Similarly, from (6), if t is in R^+ then

$$\int_{t+1}^{\infty} \| L(t, s) \| \, ds \leq \omega \Gamma^2 / q .$$

It is now clear that if t is in R^+, then

$$\int_0^{\infty} \| L(t, s) \| \, ds \leq 2\omega + 2\omega \Gamma^2 / q .$$

and it suffices to consider our theorem in the case $q = 1$.

Let M be a positive number such that (4) is true whenever t is in R^+, and let $m = 1/M$. Now differential inequality methods virtually identical to those used in establishing (5) and (6) can be used to show that

$$\| L(t, s) \| \leq \omega M e^m e^{-m|t-s|}$$

whenever (t, s) is in $R^+ \times R^+$ and $|t - s| \geq 1$. Since $e^m e^{-m|t-s|} \geq 1$ if $|t - s| \leq 1$, it follows that if $J = \max\{\omega M, 1\}$ and $\alpha = m$, we have our exponential dichotomy. This completes the proof.

REFERENCES

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