A sufficient condition for an exponential dichotomy


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RIASSUNTO. — L'Autore dà condizioni sufficienti su A(t) perché l'equazione u'(t) = f(t) + A(t) u(t) abbia soluzioni limitate.

I. INTRODUCTION AND RESULTS

Let Y be a finitedimensional linear space with norm ||, and let $\mathbb{R}^+ = [0, \infty)$. Let A be the algebra of linear functions from Y to Y, with induced norm || ||, let I be the identity in A, and let A be a locally integrable function from $\mathbb{R}^+$ to A. We propose to study the differential equations

(1) $v'(t) = A(t) v(t)$

and

(2) $u'(t) = f(t) + A(t) u(t)$

on $\mathbb{R}^+$, where f is always at least locally integrable.

Let $\Phi$ be the fundamental solution of (1), i.e., $\Phi$ is that locally absolutely continuous function from $\mathbb{R}^+$ to A such that

$$\Phi(t) = I + \int_0^t A(s) \Phi(s) \, ds$$

whenever $t$ is in $\mathbb{R}^+$, and recall that each value of $\Phi$ is invertible. Let $M_1$ be the subspace of Y to which $x$ belongs if and only if the function from $\mathbb{R}^+$ to Y described by $t \mapsto \Phi(t) x$ is bounded. Let $M_2$ be a subspace of Y such that $Y = M_1 \oplus M_2$, and let $P_1$ and $P_2$ be supplementary projections in A with ranges $M_1$ and $M_2$ respectively. Finally, let L from $\mathbb{R}^+ \times \mathbb{R}^+$ to A be given by

$L(t, s) = \Phi(t) P_1 \Phi(s)^{-1}$ if $0 \leq s \leq t$ and $L(t, s) = -\Phi(t) P_2 \Phi(s)^{-1}$ if $s > t$.

We shall say that A admits an exponential dichotomy if and only if there are positive numbers $J$ and $\alpha$ such that $\|L(t, s)\| \leq J e^{-\alpha|t-s|}$ whenever $(t, s)$ is in $\mathbb{R}^+ \times \mathbb{R}^+$. In [1], T. F. Bridgland, Jr. has shown that if there are positive numbers K and M such that

(3) $\|L(t, s)\| \leq K$

whenever \((t, s)\) is in \(R^+ \times R^+\) and
\[
\int_0^\infty \| L(t, s) \| \, ds \leq M
\]
whenever \(t\) is in \(R^+\), then \(A\) admits an exponential dichotomy. W. A. Coppel [3, Theorem 3, p. 134] has shown that (4) and the boundedness of \(A\) yield the same conclusion. The main effort of the present work is to extend Bridgland’s result to the following theorem.

**THEOREM.** Suppose that if \(c\) is a positive number then there is a positive number \(K_c\) such that \(\| L(t, s) \| \leq K_c\) whenever \((t, s)\) is in \(R^+ \times R^+\) and \(| t - s | \leq c\). Suppose also that there exists a number \(q\) of \([1, \infty)\) such that
\[
\sup_{t \geq 0} \int_0^t \| L(t, s) \|^q \, ds < \infty.
\]
Then \(A\) admits an exponential dichotomy.

Bridgland’s hypotheses are known [3, p. 131] to be equivalent to requiring that if \(f\) is in either \(L^1[R^+, Y]\) or \(L^\infty[R^+, Y]\) then (2) has bounded solution. Our present hypotheses have similar interpretations. If \(q = 1\), our second hypothesis is the same as Bridgland’s. If \(q > 1\), and \(p = q/(q - 1)\), then R. Conti [2] has shown that our second condition is equivalent to requiring that if \(f\) is in \(L^p[(R^+, Y]\) then (2) has a bounded solution. Techniques similar to those of Bridgland, Conti, and Coppel can be used to show that our first condition is equivalent to the following: If \(c\) is a positive number and \(F\) is a bounded subset of \(L^1[R^+, Y]\), each member of which has its support in an interval of length \(c\), then there is a bounded subset \(G\) of \(L^\infty[R^+, Y]\) such that if \(f\) is in \(F\) there is a solution \(u\) of (2) in \(G\).

II. PROOFS

First we shall show that it suffices to consider the case \(q = 1\), and then we shall prove the theorem in this case.

Let \(\omega\) be a positive number such that \(\| L(t, s) \| \leq \omega\) whenever \((t, s)\) is in \(R^+ \times R^+\) and \(| t - s | \leq 1\). Suppose \(q > 1\), and let \(\Gamma\) be a positive number such that
\[
\int_0^t \| L(t, s) \|^q \, ds \leq \Gamma^q
\]
whenever \(t\) is in \(R^+\). Let \(\gamma = 1/\Gamma\), and let \(\phi = q/(q - 1)\).

We now claim that if \((t, s)\) is in \(R^+ \times R^+\) and \(t - s \geq 1\) then
\[
\| L(t, s) \| \leq \omega \Gamma(t - s)^{1/\phi} \exp [\gamma q - \gamma q(t - s)^{1/\phi}].
\]
If \( P_1 = 0 \) this is obvious, so assume \( P_1 = 0 \). Let \( s \) be in \( R^+ \), and let \( \varphi \) be given on \( [s, \infty) \) by \( \varphi(t) = |L(t, s)|^{-1} \). Now, if \( t \) is in \( [s, \infty) \),

\[
\varphi(t)^{-1} \int_s^t \varphi(r) \, dr = \left( \int_s^t \varphi(r) \, dr \right) L(t, s) \leq \int_s^t \varphi(r) \cdot L(t, r) \cdot L(r, s) \, dr \\
\leq \int_s^t \|L(t, r)\| \, dr \\
\leq \left( \int_s^t \|L(t, r)\|^p \, dr \right)^{1/p} (t - s)^{1/p} \\
\leq \Gamma(t - s)^{1/p}.
\]

For convenience, put \( \sigma = s + 1 \). Now, if \( t \geq \sigma \),

\[
\Gamma(t - s)^{1/p} \varphi(t) \geq \int_s^t \varphi(r) \, dr \\
= \int_s^\sigma \varphi(r) \, dr + \int_\sigma^t \varphi(r) \, dr \\
\geq 1/\omega + \int_\sigma^t \varphi(r) \, dr,
\]

so

\[
\varphi(t) \geq (\gamma/\omega)(t - s)^{-1/p} + \gamma(t - s)^{-1/p} \int_\sigma^t \varphi(r) \, dr.
\]

Thus, if \( t \geq \sigma \), \( \varphi(t) \geq \psi(t) \), where \( \psi \) solves

\[
\psi(t) = (\gamma/\omega)(t - s)^{-1/p} + \gamma(t - s)^{-1/p} \int_\sigma^t \psi(r) \, dr
\]
on \( [\sigma, \infty) \). But \( \psi \) is given by

\[
\psi(t) = (\gamma/\omega)(t - s)^{-1/p} \exp \left[ -\gamma q + \gamma q (t - s)^{-1/q} \right],
\]

so

\[
\|L(t, s)\| = \varphi(t)^{-1} \leq \psi(t)^{-1}
\]
yields (5), and our claim is verified.
Next we claim that if $s - t \geq 1$ then

$$\| L(t, s) \| \leq \psi \Gamma(s - t)^{1/p} \exp \left[ \gamma q \gamma q (s - t)^{1/q} \right]. \tag{6}$$

If $P_2 = 0$, this is obvious, so assume $P_2 \neq 0$. Let $s$ be in $\mathbb{R}^+$, and let $\varphi$ be given on $[0, s]$ by $\varphi(t) = \| L(t, s) \|^{-1}$. As before,

$$\varphi(t)^{-1} \int_t^s \varphi(r) \, dr \leq \Gamma(s - t)^{1/p}$$

if $t$ is in $[0, s]$, so if $\sigma = s - 1$,

$$\varphi(t) \geq \left( \frac{\gamma}{\omega} \right) (s - t)^{-1/p} + \gamma (s - t)^{-1/p} \int_t^s \varphi(r) \, dr$$

whenever $t$ is in $[0, \sigma)$. This last integral inequality is somewhat unorthodox, so we shall give more detail to solving it. Let $\beta$ be given on $[0, \sigma]$ by

$$\beta(t) = \int_t^\sigma \varphi(r) \, dr.$$

Now $\beta(\sigma) = 0$, and if $t$ is in $[0, \sigma)$,

$$- \beta'(t) \geq - (\gamma/\omega) (s - t)^{-1/p} + \gamma (s - t)^{-1/p} \beta(t),$$

$$\beta'(t) \leq - (\gamma/\omega) (s - t)^{-1/p} - \gamma (s - t)^{-1/p} \beta(t),$$

$$\beta'(t) + \gamma (s - t)^{-1/p} \beta(t) \leq - (\gamma/\omega) (s - t)^{-1/p},$$

$$(\beta(t) \exp \left[ - \gamma q (s - t)^{1/q} \right]' \leq - (\gamma/\omega) (s - t)^{-1/p} \exp (- \gamma q (s - t)^{1/q}).$$

Integrating this last inequality yields

$$- \beta'(t) \exp \left[ - \gamma q (s - t)^{1/q} \right] \leq - \left( \gamma/\omega \right) \int_t^\sigma (s - r)^{-1/p} \exp \left[ \gamma q (s - r)^{1/q} \right] \, dr$$

$$= - \left( 1/\omega \right) e^{-\gamma r} + \left( 1/\omega \right) \exp \left[ - \gamma q (s - t)^{1/q} \right],$$

$$\beta(t) \geq \left( 1/\omega \right) \exp \left[ \gamma q + \gamma q (s - t)^{1/q} \right] - \left( 1/\omega \right).$$

Thus, if $t$ is in $[0, \sigma)$,

$$\varphi(t) \geq \left( \gamma/\omega \right) (s - t)^{-1/p} + \gamma (s - t)^{-1/p} \beta(t)$$

$$\geq \left( \gamma/\omega \right) (s - t)^{-1/p} \exp \left[ - \gamma q + \gamma q (s - t)^{1/q} \right],$$

(6) follows, and our second claim is verified.
Suppose that \( t \geq 1 \). From (5) we have
\[
\int_0^{t-1} \| L(t, s) \| \, ds \leq \omega \Gamma^2 \exp \left( -\gamma q (t - s)^{1/q} \right) ds
\]
\[= (\omega \Gamma^2/q) e^{\gamma q} (e^{\gamma q} - \exp (-\gamma q t^{1/q})) \leq \omega \Gamma^2/q.
\]
Similarly, from (6), if \( t \) is in \( \mathbb{R}^+ \) then
\[
\int_{t+1}^{\infty} \| L(t, s) \| \, ds \leq \omega \Gamma^2/q.
\]
It is now clear that if \( t \) is in \( \mathbb{R}^+ \), then
\[
\int_0^{\infty} \| L(t, s) \| \, ds \leq 2 \omega + 2 \omega \Gamma^2/q.
\]
and it suffices to consider our theorem in the case \( q = 1 \).

Let \( M \) be a positive number such that (4) is true whenever \( t \) is in \( \mathbb{R}^+ \), and let \( m = 1/M \). Now differential inequality methods virtually identical to those used in establishing (5) and (6) can be used to show that
\[
\| L(t, s) \| \leq \omega Me^{-m|t-s|}
\]
whenever \( (t, s) \) is in \( \mathbb{R}^+ \times \mathbb{R}^+ \) and \( |t - s| \geq 1 \). Since \( e^m e^{-m|t-s|} \geq 1 \) if \( |t - s| \leq 1 \), it follows that if \( J = \max \{ \omega M, 1 \} \) and \( \alpha = m \), we have our exponential dichotomy. This completes the proof.

References