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Summability factor on Abel type methods

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Analisi. — *Summability factor on Abel type methods.* Nota di
BABBAN PRASAD MISHRA e DINESH SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Estensione al metodo di Abel dei risultati ottenuti da Bosanquet per la sommabilità di serie.

1. INTRODUCTION

Suppose that $\{s_n\}$, $n = 0, 1, \cdots$ be any sequence of numbers and that, for $-1 < \lambda$,

$$(1.1)\qquad E_n^\lambda = \binom{n+\lambda}{n}, \qquad \text{for } n = +1, +2, \cdots,$$

$$E_0^\lambda = 1 \quad , \quad E_n^\lambda = 0 , \qquad \text{for } n = -1, -2, \cdots,$$

$$s_n = \sum_{v=0}^{n} a_v \quad , \quad t_n = \sum_{v=0}^{n}{}' a_v \, \varepsilon_v$$

$$(1.2)\qquad \varphi_\lambda(x) = (1-x)^{\lambda+1} \sum E_n^\lambda s_n x^n,$$

$$\varphi_{\lambda,\varepsilon}(x) = (1-x)^{\lambda+1} \sum E_n^\lambda t_n x^n.$$

Clearly

$$\varphi_{\lambda,1}(x) = \varphi_\lambda(x) \quad , \quad \varphi_0(x) = \varphi(x) = \sum a_n x^n.$$

With Borwein [1], we say that the sequence $\{s_n\}$ is summable (A_λ) to s and write $s_n \to s\,(A_\lambda)$, if the series defining $\varphi_\lambda(x)$ is convergent for all x in the open interval $(0, 1)$ and tends to the finite limit s as $x \to 1-0$. The (A_0) method is the ordinary Abel method.

We define S_n^α, for $n \geq 0$, by

$$(1.4)\qquad S_n^\alpha = S_n^\alpha(s_v) = \sum_{v=0}^{n}{}' E_{n-v}^{\alpha-1} s_v , \qquad \alpha \text{ is an integer.}$$

Thus

$$S_n^1 = s_0 + s_1 + \cdots + s_n \quad , \quad S_n^0 = s_n \quad , \quad S_n^{-1} = a_n ,$$

and

$$(1.5)\qquad E_n^\alpha = S_n^\alpha(1) .$$

(*) Nella seduta del 14 dicembre 1974.

For positive integer α, we have

$$\text{(1.6)} \qquad S_n^{-\alpha}(s_v) = \sum_{\mu=0}^{\alpha} (-)^{\mu} \binom{\alpha}{\mu} (\alpha)\, s_{n-\mu} \qquad (n \geq \alpha),$$

since $E_n^{-\alpha} = 0$ for $n \geq \alpha$.

If we write

$$\text{(1.7)} \qquad \Delta^0 u_n = u_n \quad , \quad \Delta' u_n = u_n - u_{n+1} \quad , \quad \Delta^k = \Delta(\Delta^{k-1}),$$

then

$$\text{(1.8)} \qquad \Delta^k s_n = (-)^k S_{n+k}^{-k}(s_v).$$

We shall use the following identities which will be used in the sequel.

$$\text{(1.9)} \qquad S_n^{\alpha+\beta} = S_n^{\beta}(S_v^{\alpha}),$$

where α, β are zero or integers

$$\text{(1.10)} \qquad \sum E_n^{\alpha'} x^n = (1 - x)^{-\alpha'-1} \qquad (|x| < 1, \alpha' > -1).$$

2. THE FOLLOWING THEOREMS ARE KNOWN

THEOREM A. *If* $-1 \leq \rho \leq k$ *(ρ, k integers) and p is real, then necessary and sufficient conditions for $\{t_n\}$ to be summable* (C, ρ) *whenever* $S_n^k = O(n^{k+p})$ *are*

$$\text{(2.1)} \qquad \varepsilon_n = o(n^{\rho-k-p}) \quad , \quad \sum (n+1)^{k+p} |\Delta^{k+1} \varepsilon_n| < \infty.$$

THEOREM B. *If* $-1 \leq k$, *p is real,* $-1 < \lambda$ *and the sequence* $\{t_n\}$ *is summable* (A_λ) *whenever* $S_n^k = o(n^{k+p})$, *then*

$$\text{(2.2)} \qquad \sum (n+1)^{k+p} |\Delta^{k+1}\{\varepsilon_n E_n^{\lambda} f(\lambda, n)\}| < \infty,$$

where

$$f(\lambda, n) = 1 - \frac{\lambda}{\underline{|1}} \frac{n}{n+1} + \frac{\lambda(\lambda-1)}{\underline{|2}} \frac{n}{n+2} - \frac{\lambda(\lambda-1)(\lambda-2)}{\underline{|3}} \frac{n}{n+3} \cdots$$

Theorems A and B are due respectively to Bosanquet [2], Mishra and Singh [4]. In this paper, we will prove the following theorem.

THEOREM C. *If p is real,* $-1 < \lambda$, *k is an integer such that* $-1 \leq k$ *then necessary and sufficient conditions for the sequence* $\{t_n\}$ *to be summable* (A_λ) *whenever* $S_n^k = O(n^{k+p})$, *are*

$$\text{(2.3)} \qquad \varepsilon_n = o(n^{-p-\lambda})$$

and

$$\text{(2.4)} \qquad \sum (n+1)^{k+p} |\Delta^{k+1}\{\varepsilon_n E_n^{\lambda} f(\lambda, n)\}| < \infty.$$

3. The lemmas, needed for the proof, are collected below

LEMMA 1. *If* $p \neq 1, -2, \cdots, -k$, *then* $S_n^k = O(n^{k+p})$ *whenever* $s_n = O(n^p)$

LEMMA 2. *If* q *is real and* $\sum (n+1)^q |\Delta \varepsilon_n| < \infty$, *then there is a number* s *such that* $\varepsilon_n = s + o(n^{-q})$.

LEMMA 3. *If the sequence* $\{s_n\}$ *is summable* (A_λ) *and* $a_n = O(n^{-1})$ *then the sequence* $\{s_n\}$ *is convergent.*

For Lemmas 1 and 2, see Bosanquet [2] while Lemma 3 is due to Mishra [3].

4. In this section, we give the proof of theorem C

Sufficiency. By virtue of condition (2.3), it is easy to show that the series defining $\varphi_{\lambda,\varepsilon}(x)$ is convergent for $0 \leq x < 1$, whenever $S_n^k = O(n^{k+p})$. As the general term of a convergent series tends to zero, we have, after $k+1$ partial summations

$$\varphi_{\lambda,\varepsilon}(x) = (1-x)^{\lambda+1} \sum S_n^k \Delta^{k+1} \left\{ \varepsilon_n \sum_{v=n}^{\infty} E_v^\lambda x^v \right\} \tag{4.1}$$

for $0 \leq x < 1$. In particular if we take

$$S_n^k = (n+1)^{k+p},$$

then

$$\varphi_{\lambda,\varepsilon}(x) = (1-x)^{\lambda+1} \sum (n+1)^{k+p} \Delta^{k+1} \left\{ \varepsilon_n \sum_{v=n}^{\infty} E_v^\lambda x^v \right\}. \tag{4.2}$$

Making $x \to 1-0$ and using condition (2.4), we see that

$$\lim_{x\to 1-0} \varphi_{\lambda,\varepsilon}(x) = \lim_{x\to 1-0} \sum (n+1)^{k+p} \Delta^{k+1} \{ \varepsilon_n E_n^\lambda x^n f(\lambda, n, x) \},$$

where

$$f(\lambda, n, x) = 1 - \frac{\lambda}{\underline{|1}} \frac{n}{n+1} x + \frac{\lambda(\lambda-1)}{\underline{|2}} \frac{n}{n+2} x^2 - \frac{\lambda(\lambda-1)(\lambda-2)}{\underline{|3}} \frac{n}{n+3} x^3 \cdots.$$

Clearly

$$\lim_{x\to 1-0} f(\lambda, n, x) = f(\lambda, n).$$

Hence

$$\lim_{x\to 1-0} \varphi_{\lambda,\varepsilon}(x) = \sum (n+1)^{k+p} \Delta^{k+1} \{ \varepsilon_n E_n^\lambda f(\lambda, n) \}. \tag{4.3}$$

Condition (4.3), since (2.4) holds, ensures the truth of sufficiency part of the theorem.

Necessity. $(A_{\lambda'})$ summability $(0<\lambda')$ implies (A_λ) summability $(-1<\lambda<0)$. Therefore we need to consider only the case $-1<\lambda<0$. Necessity of (2.4) follows from theorem B. We divide the proof of (2.3) into the following cases:

Case 1.

$$p \neq -1, -2, -3, \cdots, -k.$$

Case 2.

$$p = -1, -2, \cdots, -k.$$

Proof of Case 1. After Lemma 1, if follows that the sequence $\{t_n\}$ is summable (A_λ) whenever $s_n = O(n^p)$ and thus by Theorem B, with $k = 0$,

$$\sum (n+1)^p \, |\Delta\{\varepsilon_n \, E_n^\lambda f(\lambda, n)\}| < \infty. \tag{4.4}$$

By virtue of Lemma 2, we obtain

$$\varepsilon_n \, E_n^\lambda f(\lambda, n) = o(n^{-p}) + s, \tag{4.5}$$

where s is a constant.

In the case $p<0$, s is arbitrary and therefore it may be taken as zero.

For $p \geq 0$, we observe that $\varepsilon_n - s\{E_n^\lambda f(\lambda, n)\}^{-1}$, satisfies the conditions of ε_n in sufficiency part of Theorem C. Hence the series $\sum a_n\{\varepsilon_n - s(E_n^\lambda f(\lambda, n))^{-1}\}$ is summable (A_λ) whenever $S_n^k = O(n^{k+p})$. The (A_λ) summability of the series $\sum a_n \varepsilon_n$ implies the (A_λ)-summability of $\sum s a_n\{E_n^\lambda f(\lambda, n)\}^{-1}$.

If $p>0$ and $a_n = (n+1)^{p-1}\{E_n^\lambda f(\lambda, n)\}$ then clearly $S_n^k = O(n^{k+p})$ and $\sum s a_n\{E_n^\lambda f(\lambda, n)\}^{-1}$ is not summable (A_λ). Therefore s is necessarily zero.

Now consider the case $p=0$ and take as an example

$$a_n\{E_n^\lambda f(\lambda, n)\}^{-1} = (n+1)^{-1}\cos\{\log(n+1)\}. \tag{4.6}$$

This example is due to Bosanquet [2]. By an argument similar to that of Theorem A, we can prove, after Lemma 3, that $\sum s a_n\{E_n^\lambda f(\lambda, n)\}^{-1}$ is not summable (A_λ) whenever $S_n^k = O(n^{k+p})$ and so also in this case $s = 0$. This proves Case 1.

Proof of Case 2. Suppose $p = -\mu$, where $\mu = 1, 2, \cdots, k$. The sequence $\{t_n\}$ is summable (A_λ) whenever $S_n^k = O(n^{k-\mu})$. By virtue of (1.9) with $\beta = k-\mu$ and $\alpha = \mu$, we see that $S_n^\mu = O(1)$ implies $S_n^k = O(n^{k-\mu})$. Thus the sequence $\{t_n\}$ is summable (A_λ) whenever $S_n^\mu = O(1)$. If follows from Theorem B, with $k = -p = \mu$, that

$$\sum |\Delta^{\mu+1}\{\varepsilon_n \, E_n^\lambda f(\lambda, n)\}| < \infty. \tag{4.7}$$

We have, after (1.7) and (1.8) and Lemma 2,

$$S_n^{-\mu}\{\varepsilon_v \, E_v^\lambda f(\lambda, v)\} = s + o(1). \tag{4.8}$$

Repeated applications of (1.9) with $\beta = 1$ and (1.5) gives

$$\varepsilon_n \, E_n^\lambda f(\lambda\,,n) = s E_n^\mu + o(n^\mu)\,. \tag{4.9}$$

Since $\Delta^{\mu+1} E_n^\mu = (-)^{\mu+1} E_{n+\mu+1}^{-1} = 0$ for $n \geq 0$, it follows from (4.7) and (4.9) that $\varepsilon_n - s E_n^\mu \{E_n^\lambda f(\lambda\,,n)\}^{-1}$ satisfies the conditions of ε_n in sufficiency part of the theorem with $k = -p = \mu$. Hence the series $\sum a_n \{s E_n^\mu (E_n^\lambda f(\lambda\,,n))^{-1}\}$ is summable (A_λ) whenever $S_n^\mu = O(1)$. If we consider

$$\{E_n^\lambda f(\lambda\,,n)\}^{-1} a_n = (n+1)^{-\mu-1} \cos\{\log(n+1)\} \tag{4.10}$$

then clearly the series $\sum s a_n E_n^\mu \{E_n^\lambda f(\lambda\,,n)\}^{-1}$ is not summable (A_λ) whenever $S_n^\mu = O(1)$ and so also in this case $s = 0$. This completes the proof of the necessary part of the theorem.

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