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On the convergence of sequence of Iterates

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Analisi funzionale. — *On the convergence of sequence of Iterates.* Nota di S. P. SINGH, presentata (*) dal Corrisp. G. FICHERA.

RIASSUNTO. — In un recente lavoro Diaz e Metcalf hanno provato il seguente teorema. Sia C un sottoinsieme chiuso e convesso di uno spazio di Banach X strettamente convesso. Sia $T : C \to C$ una trasformazione non espansiva e compatta. Allora, per qui $x \in C$, la successione $\{T_\lambda^n x\}$ (dove $T_\lambda : C \to C$ è così definita $T_\lambda(x) = \lambda Tx + (1 - \lambda)x$, $x \in C$, $0 < \lambda < 1$) converge ad un punto fisso di T. In questa Nota abbiamo esteso il risultato alle trasformazioni densificanti e abbiamo dato alcuni corollari.

A sequence of iterates $\{T^n x_0\}$ of a non expansive mapping T need not converge to a fixed point of T. Therefore one considers a mapping of the form

$$T_\lambda = \lambda T + (1 - \lambda) I , \qquad 0 < \lambda < 1,$$

and then the sequence of iterates $\{T_\lambda^n x_0\}$, under suitable restrictions, converges to a fixed point of T.

Diaz and Metcalf [1] proved the following well known theorem.

Let X be a strictly convex Banach space, C a closed, bounded, convex subset of X and let $T : C \to C$ be a non-expansive mapping, and suppose T (C) is contained in a compact subset C_1 of C. Then, for $x_0 \in C$, the sequence of iterates $\{T_\lambda^n x_0\}$, where $T_\lambda : C \to C$ is a mapping defined by $T_\lambda = \lambda T + (1 - \lambda) I$, $0 < \lambda < 1$, converges to a fixed point of T.

The aim of this Note is to prove the convergence result for a more general type of mapping introduced by Hardy and Rogers [4].

The mapping T is defined on X such that

$$\text{(A)} \qquad \| Tx - Ty \| \leq a_1 \| x - y \| + a_2 (\| x - Tx \| + \| y - Ty \|) + a_3 (\| y - Tx \| + \| x - Ty \|)$$

where $a_i \geq 0$, $i = 1, 2, 3$ and $a_1 + 2a_2 + 2a_3 \leq 1$.

We prove the following theorem.

Let C be a closed, bounded, convex subset of a strictly convex Banach space X. Let $T : C \to C$ be a densifying mapping satisfying condition (A). Then, for $x_0 \in C$, the sequence of iterates $\{T_\lambda^n x_0\}$, where $T_\lambda : C \to C$ is a mapping defined by $T_\lambda = \lambda T + (1 - \lambda) I$, $0 < \lambda < 1$; converges to a fixed point of T.

We need the following preliminaries:

Let A be a bounded subset of X. The measure of non-compactness of A, denoted by $\alpha(A)$, is defined to be the infimum of $\varepsilon > 0$ such that A can be covered by a finite subset with diameter $< \varepsilon$.

(*) Nella seduta del 14 novembre 1974.

It is known [5] that,

1) $\alpha(A) = 0 \Longleftrightarrow A$ is precompact.

2) $\alpha(A \cup B) \leq \max\{\alpha(A), \alpha(B)\}$.

3) $\alpha(A) = \alpha(\bar{A})$; $\bar{A}$ stands for the closure of A.

4) $\alpha(A + B) = \alpha(A) + \alpha(B)$, where A and B are subsets of X.

A continuous mapping $T : X \to X$ is called densifying, if for any bounded set A with $\alpha(A) > 0$, we have

$$\alpha(TA) < \alpha(A).$$

A contraction mapping and a completely continuous mapping are examples of densifying mappings.

The following known results will be required in the proof.

B_1 : Let $T : C \to C$ be a densifying mapping defined on a closed, bounded, convex subset of a Banach space X. Then T has at least a fixed point in C, [3].

B_2 : Let $T : X \to X$ be a continuous mapping.

Suppose

(i) $F(T) \neq \varphi$, where $F(T)$ is the set of fixed points of T.

(ii) for each $y \in X$, $y \notin F(T)$ and each $u \in F(T)$.

$$\| Ty - u \| < \| y - u \|.$$

Let $x_0 \in X$. Then, either $\{T^n x_0\}$ has no convergent subsequence or $\{T^n x_0\}$ converges to a fixed point of T [1].

Proof. Since T is densifying on C, T_λ is also densifying. Indeed, let A be a bounded subset of C.

Then

$$T_\lambda A = \lambda TA + (1 - \lambda) A.$$

Now

$$\begin{aligned} \alpha(T_\lambda A) &\leq \lambda\alpha(TA) + (1 - \lambda)\alpha(A) \\ &< \lambda\alpha(A) + (1 - \lambda)\alpha(A) \quad \text{(since T is densifying)} \\ &= \alpha(A). \end{aligned}$$

Moreover $F(T_\lambda) \neq \varphi$, since $F(T) = F(T_\lambda)$ and $F(T) \neq \varphi$ by B_1.

Let

$$D = \bigcup_{n=0}^{\infty} T_\lambda^n x_0.$$

Then

$$T_\lambda D = \bigcup_{n=1}^{\infty} T_\lambda^n x_0.$$

Thus

$$T_\lambda D \subset D,$$

and

$$D = \{x_0\} \cup T_\lambda D.$$

Let $\bar{D}$ stand for the closure of D. Then

$$T_\lambda \bar{D} \subset \overline{T_\lambda D} \subset \bar{D},$$

i.e. $\bar{D}$ is invariant under T_λ.

We now prove that $\bar{D}$ is compact. It suffices to prove that $\alpha(D) = 0$, since X complete, $\bar{D}$ precompact implies that $\bar{D}$ is compact.

Let us assume that $\alpha(D) > 0$.

Since $D = \{x_0\} \cup T_\lambda D$

$$\begin{aligned}\alpha(D) &= \max\{\alpha(x_0), \alpha(T_\lambda(D))\} \\ &= \max\{0, \alpha(T_\lambda D)\} \\ &= \alpha(T_\lambda(D)),\end{aligned}$$

a contradiction to the fact that T_λ is densifying.

Hence, $\alpha(D) = 0$, i.e. $\alpha(\bar{D}) = 0$. Thus $\bar{D}$ is compact. Hence the sequence of iterates $\{T_\lambda^n x_0\}$ has a convergent subsequence.

In order to apply B_2, it only remains to show that for each $x \notin F(T_\lambda)$ and $u \in F(T_\lambda)$,

$$\| T_\lambda x - u \| < \| x - u \| .$$

Now, clearly,

$$\| Tx - u \| \leq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} \| x - u \| ,$$

just by using condition (A) and the triangle inequality.

Thus

$$\| Tx - u \| \leq \| x - u \| , \qquad (p)$$

In case $x \neq u$, we have

$$\begin{aligned}\| T_\lambda x - u \| &= \| T_\lambda x - T_\lambda u \| \\ &= \| \lambda (Tx - u) + (1 - \lambda)(x - u) \| \\ &= \| x - u \| \left\| \frac{\lambda (Tx - u)}{\| x - u \|} + \frac{(1 - \lambda)(x - u)}{\| x - u \|} \right\| .\end{aligned}$$

If strict inequality holds in (p), we get

$$\begin{aligned}&\left\| \frac{\lambda (Tx - u)}{\| x - u \|} + \frac{(1 - \lambda)(x - u)}{\| x - u \|} \right\| \\ &\leq \lambda \frac{\| Tx - u \|}{\| x - u \|} + (1 - \lambda) \\ &< 1 .\end{aligned}$$

Hence,

$$\| T_\lambda x - u \| < \| x - u \| .$$

However, if equality sign holds, then, since $\frac{(Tx - u)}{\| x - u \|}$, and $\frac{(x - u)}{\| x - u \|}$ have unit norm and X is strictly convex Banach space, we get

$$\left\| \frac{\lambda (Tx - u)}{\| x - u \|} + (1 - \lambda) \frac{(x - u)}{\| x - u \|} \right\| < 1 .$$

Thus

$$\| T_\lambda x - u \| < \| x - u \| \quad , \quad x \neq u .$$

Thus the proof.

1) In case $a_2 = a_3 = 0$ and $a_1 = 1$ we get a result due to Diaz and Metcalf [1], since every completely continuous map is densifying. If $\lambda = 1/2$, we get a result due to Edelstein [2].

2) In case $a_2 = a_3 = 0$ amd $a_1 = 1$ we get a result due to Petryshyn [6].

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