Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Olusola Akinyele

A multiplier problem

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.6, p. 487–490.

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<http://www.bdim.eu/item?id=RLINA_1974_8_57_6_487_0>

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Matematica. — *A multiplier problem.* Nota di OLUSOLA AKINYELE, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Sia G un gruppo abeliano localmente compatto e $\hat{G}$ il suo duale. I moltiplicatori di $L_1(G)$ sono stati identificati con l'algebra di misura M (G). Sia A un'algebra commutativa di Banach. Indichiamo con $B^1(G, A)$ lo spazio delle funzioni integrabili secondo Bochner rispetto alla misura di Haar di G. Proviamo il seguente teorema: Un operatore lineare limitato T su $B^1(G, A)$ in sè è un moltiplicatore se e solo se esiste una unica misura vettoriale m definita su G e tale che $Tf = m * f$ per ogni $f \in B^1(G, A)$,

§ 1. INTRODUCTION

Let G be a locally compact Abelian group and A commutative Banach algebra. Denote by $B^1(G, A)$, the Bochner integrable functions with respect to the Haar measure of G [cfr. 2]. Let M (G) denote the measure algebra of the locally compact group G and $C_0(G, A)$ the space of continuous functions vanishing at infinity and with values in A. In this paper we prove that if a bounded linear operator T commutes with translations on $C_0(G, A)$ then there exists a unique vector-valued measure m defined on G such that $Tf = m * f$ for each $f \in C_0(G, A)$. Moreover we show that a bounded linear operator T is a multiplier of $B^1(G, A)$ if and only if there exists a unique vector-valued measure m defined on G such that $Tf = m * f$ for every $f \in B^1(G, A)$.

§ 2. PRELIMINARIES

The Fourier transform $\hat{f}$ of $f \in B^1(G, A)$ is defined by:

$$\hat{f}(\chi, M) = \int_G \varphi_M(f(x))\, \chi(x)\, d\nu(x) \qquad \chi \in \hat{G}$$

where φ_M is a complex-valued homomorphism of A [cf. 2], $\hat{G}$ is the character group of G and ν is the Haar measure of G.

DEFINITION 2.1. Let $T : B^1(G, A) \to B^1(G, A)$ be a bounded linear operator. T will be called a multiplier if and only if $T(f * g) = Tf * g$ for $f, g \in B^1(G, A)$.

Let E and F be two normed vector spaces. Following the notations of [1], we denote by $\mathscr{L}(E, F)$ the vector space of linear continuous operators $U : E \to F$, equipped with the norm $\|U\| = \operatorname{Sup}\{\|Ux\| ; \|x\| \leq 1, x \in E\}$.

(*) Nella seduta del 14 dicembre 1974.

If E, F are both Banach algebras with $E = F$, then $\mathscr{L}(E, F)$ becomes $\mathscr{L}^*(E)$ the Banach algebra of continuous operators on E. It is well known that there exists an isometric and isomorphic embedding of E into $\mathscr{L}^*(E)$. Moreover if E is the space of complex numbers C, then for every Banach space F, $\mathscr{L}^*(E, F) = F$.

Set $E=F=A$ and denote by $M(G, \mathscr{L}^*(A))$ the Banach space of all regular Borel vector measures on G, with finite variation. For $m \in M(G, \mathscr{L}^*(A))$ with finite variation $\mu |m| (G) = \mu(G) = \|\mu\|$, where $\|\cdot\|$ is the norm of M(G). Suppose we take A to be equal to the space of complex numbers then $\mathscr{L}^*(C, C) = C$ and $M(G, \mathscr{L}^*(A))$ thus specializes into M(G).

Denote by $C_{00}(G, A)$, the space of continuous functions $f : G \to A$, with compact support. We consider on $C_{00}(G, A)$ the topology of uniform convergence defined by the norm $\|f\| = \sup_{x \in G} |f(x)|$.

Let A be a commutative Banach algebra and let $(u, v) \to uv$ be a bilinear mapping of $\mathscr{L}^*(A) \times A$ into A such that $\|uv\| \leq \|u\| \|v\|$.

Suppose $m \in M(G, \mathscr{L}^*(A))$, then we can integrate with respect to m, functions $f : G \to A$ and $\int_G f(x) \, dm(x) \in A$ [cf. 1, § 7].

Let $m \in M(G, \mathscr{L}^*(A))$ with finite variation μ and $f \in C_{00}(G, A)$, then $\int_G f(x) \, dm(x)$ belongs to A. Suppose a function g is defined by $g(x, y) = f(xy^{-1})$, $x, y \in G$, $f \in C_{00}(G, A)$. Then $g \in C_{00}(G \times G, A)$ and $\int_G g(x, y) \, dm(y)$ exists as a continuous function of x with values in A. Define $m * f$ on G, by setting;

$$(m * f)(x) = \int_G f(xy^{-1}) \, dm(y) \qquad \text{for} \quad f \in C_{00}(G, A) \qquad \text{and} \quad x \in G.$$

Clearly the restriction that $f \in C_{00}(G, A)$ can be replaced by the more general condition that $f \in B^1(G, A)$; and we have,

$$(m * f)(x) = \int_G f(xy^{-1}) \, dm(y) \qquad \text{for} \quad f \in B^1(G, A), \qquad x \in G.$$

LEMMA 2.1. *Suppose* $m \in M(G, \mathscr{L}^*(A))$ *with finite variation* μ *and* $f \in B^1(G, A)$, *then* $m * f \in B^1(G, A)$.

Proof. With $f \in C_{00}(G, A)$, $m * f \in B^1(G, A)$ and since $C_{00}(G, A)$ is dense in $B^1(G, A)$, $m * f \in B^1(G, A)$ for $f \in B^1(G, A)$.

LEMMA 2.2. *For* $g \in B^1(G, A)$ *and* $x \in G$, $\hat{g}_x(\chi, M) = \chi(x) \hat{g}(\chi, M)$, *where* g_x *is a function defined by* $g_x(y) = g(yx^{-1})$, *for each* $y \in G$.

The proof is straightforward.

§ 3. MULTIPLIERS OF $B^1(G, A)$

In this section, we prove the main theorem of this paper. The translation operators τ_x are defined for functions f in $B^1(G, A)$, by the formulae $\tau_x f(y) = f(x^{-1} y)$, $x, y \in G$. It is obvious that $\|\tau_x f\| = \|f\|$.

THEOREM 3.1. *Suppose that* $T : C_0(G, A) \to C_0(G, A)$ *commutes with translation operators* τ_x (i.e. $\tau_x T = T\tau_x$ *for* $x \in G$). *Then* $\exists$ *a unique* $m \in M(G, \mathscr{L}^*(A))$ *such that* $Tf = m * f$ *for every* $f \in C_0(G, A)$.

Proof. Let e be the identity element of G. Define a mapping $F : C_{00}(G, A) \to A$ by setting $F(f) = Tf(e)$. Linearity and continuity of T from $C_0(G, A)$ into itself implies $\exists$ a number $k \geq 0$ such that $\|Tf(e)\| \leq k \|f\|_\infty$ for $f \in C_0(G, A)$. Hence F is a bounded linear operator and by Theorem 2, § 19 of [1], $\exists$ a unique $\lambda \in M(G, \mathscr{L}^*(A))$ such that $F(f) = \int_G f(y)\, d\lambda(y)$ for each $f \in C_{00}(G, A)$. In other words,

$$Tf(e) = \int_G f(y)\, d\lambda(y), \qquad \text{for} \quad f \in C_{00}(G, A).$$

Choose $x \in G$, then since $\tau_x T = T\tau_x\ x \in G$,

$$\begin{aligned} Tf(x) = (\tau_{x^{-1}} Tf)(e) = T\tau_{x^{-1}} f(e) &= \int_G (\tau_{x^{-1}} f)(y)\, d\lambda(y) \\ &= \int_G f(xy)\, d\lambda(y) \\ &= \int_G f(xy^{-1})\, d\check{\lambda}(y) = (\check{\lambda} * f)(x) \\ &= (m * f)(x), \end{aligned}$$

where $m = \check{\lambda} \in M(G), \mathscr{L}^*(A))$ is the measure defined by

$$\int_G \varphi(y)\, dm(y) = \int_G \varphi(y^{-1})\, d\lambda(y), \qquad \text{for} \quad \varphi \in C_{00}(G, A).$$

Hence $Tf = m * f$ for $f \in C_{00}(G, A)$. Since $C_{00}(G, A)$ is dense in $C_0(G, A)$, the linear continuous operator T can be extended uniquely to a continuous linear mapping of $C_0(G, A)$ into $C_0(G, A)$, denoted also by T and we still have $Tf = m * f$ for each $f \in C_0(G, A)$.

THEOREM 3.2. *Lef* G *be a locally compact Abelian group and* A *a commutative semi-simple Banach algebra. Suppose* $T : B^1(G, A) \to B^1(G, A)$

is a bounded linear operator. Then the following are equivalent:

(i) T *is a multiplier of* $B^1(G, A)$,

(ii) $(\tau_x T)(f) = (T\tau_x)(f)$ *for each* $x \in G$ *and* $f \in B^1(G, A)$,

(iii) *There exists a unique* $m \in M(G, \mathscr{L}^*(A))$ *such that* $Tf = m * f$ *for each* $f \in B^1(G, A)$.

Proof. Suppose (i) holds. Then for $f, g \in B^1(G, A)$, $T(f * g) = Tf * g$. Fix $f \in B^1(G, A)$ and vary $g \in B^1(G, A)$. Then using Lemma 2.2,

$$\overline{\widehat{(\tau_x Tf)} * g} = \overline{\tau_x \widehat{Tf}} \, \hat{g} = \chi(x) \, \widehat{Tf} \, \hat{g} = \widehat{Tf} \, \widehat{\tau_x g} = \overline{T \widehat{(f * \tau_x g)}} = T \, \overline{\tau_x \widehat{f} * g}.$$

Since $B^1(G, A)$ is semi-simple, $(\tau_x Tf) * g = (T\tau_x f) * g$ for each $g \in B^1(G, A)$, It follows that $\tau_x T = T\tau_x$ for $x \in G$ and so (i) implies (ii). Assume (ii), then by Theorem 3.1, $\exists$ a unique $m \in M(G, \mathscr{L}^*(A))$ such that $Tf = m * f$ for each $f \in C_0(G, A)$. Since $C_0(G, A)$ is dense in $B^1(G, A)$ we see that $Tf = m * f$ for each $f \in B^1(G, A)$. So (ii) implies (iii). That (iii) implies (i) is quite obvious.

Remark. If we take $A = C$, the space of complex numbers, then $M(G), \mathscr{L}^*(A))$ becomes $M(G)$, $B^1(G, A)$ becomes $L^1(G)$ and $C_0(G, A)$, $C_{00}(G, A)$ become $C_0(G)$ and $C_{00}(G)$ respectively. In that case, Theorems 3.1. and 3.2. are well known for $C_0(G)$ and $L^1(G)$ respectively [cfr. 3].

References

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