JOAO B. PROLLA

On Polynomial Algebras of Continuously Differentiable Functions


Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_6_481_0>

Riassunto. — Sia E uno spazio di Hilbert reale e separabile e sia F uno spazio di Banach reale. Viene esteso il teorema di Nachbin sulla densità delle algebre di funzioni di classe \( C^m \) a certe algebre polinomiali di funzioni da E ad F.

§ 1. INTRODUCTION

Let E and F be two real Banach spaces, with \( F = \{0\} \). Then \( C^m(E; F) \) denotes the vector space of all maps \( f : E \to F \) which are of class \( C^m \). We shall introduce two topologies on \( C^m(E; F) \). The first one is the topology \( \tau_u \) of uniform convergence of the functions and their derivatives on the compact subsets of E. It may be defined by the family of seminorms of the form

\[
P_k(f) = \max \left\{ \sup \{ \| D^k f(x) \| ; x \in K \} ; 0 \leq k \leq m \right\}
\]

where \( K \) is a compact subset of E. The second topology, denoted by \( \tau_c \), is defined by the family of seminorms of the form

\[
P_{K,L}(f) = \max \left\{ \sup \{ \| D^k f(x) \| (v) ; x \in K, v \in L \} ; 0 \leq k \leq m \right\},
\]

where \( K \) and \( L \) are compact subsets of E and \( T^*(v) = T(v, \cdots, v) \), when \( T \in \mathcal{L}_c^{(k)}(E; F) \), \( 0 \leq k \leq m \).

If \( F = \mathbb{R} \), then \( C^m(E; \mathbb{R}) \) is an algebra, denoted simply by \( C^m(E) \). If \( E = \mathbb{R}^n \) and \( F = \mathbb{R} \), Nachbin proved in [4] an analogue of the Stone-Weierstrass theorem for the topology \( \tau_u \). In fact, he gave necessary and sufficient conditions for a subalgebra of \( C^m(V) \) to be \( \tau_u \)-dense, where \( V \) is an \( n \)-dimen-


sional $C^m$-manifold, $m \geq 1$. If $E$ is a real, separable Hilbert space and $F = \mathbb{R}$, J. Lesmes gave in [3] sufficient conditions for a subalgebra of $C^1(E)$ to be $\tau_u$-dense.

In the case of a general $F$ the space $C^m(E; F)$ is not an algebra. However, we can still get a Stone-Weierstrass theorem for the so called polynomial algebras. For each integer $n \geq 1$, $P_n(E; F)$ denotes the vector subspace of $C(E; F)$ generated by the set of all maps of the form $x \mapsto u^n(x)^*u$, where $u^* \in E^*$, the topological dual of $E$, and $u \in F$. The elements of $P_n(E; F)$ are called $n$-homogeneous continuous polynomials of finite type from $E$ into $F$. The vector subspace generated by the union of all $P_n(E; F)$, $n \geq 1$, and the constant maps, is denoted by $P(E; F)$. A vector subspace $AC(C(E; F))$ is called a polynomial algebra if, given $g \in A$ and $\rho \in P_f(E; F)$, where $n \geq 1$, then $\rho \circ g$ belongs to $A$. In our joint work [5] with S. Machado we proved that the Stone-Weierstrass theorem is true for polynomial algebras of continuous functions.

In this paper, we first extend Nachbin's theorem for polynomial algebras in the case in which $E$ is finite dimensional, $F$ is any real Banach space, and $C^m(E; F)$ has the $\tau_u$ topology. Afterwards, we apply this result to the case in which $E$ is a real, separable Hilbert space, to obtain sufficient conditions for a polynomial algebra to be dense in $C^m(E; F)$ with the $\tau_u$ topology. As an application we show that, if $\Phi : E \to F$ is a $C^1$-isomorphism of $E$ onto some open subset (e.g. the open unit ball) of $F$, then the $F$-valued polynomials in $\Phi$ are $\tau_u$-dense in $C^1(E; F)$, and real valued polynomials in $\Phi$ are $\tau_u$-dense in $C^1(E)$, when $F$ is another real separable Hilbert space.

This work was done while visiting the Institut für Angewandte Mathematik u. Informatik der Universität Bonn, by invitation of the "Sonderforschungsbereich 72" (Teilprojekt A3, Approximationsverfahren in metrischen Räumen), to whose members the Author thanks the hospitality.

§ 2. Vector-valued functions of $n$ variables

In this section $E$ is a finite-dimensional real Banach space and $F$ is any real Banach space, not reduced to $\{0\}$. Since $E$ is $C^\infty$-isomorphic with $\mathbb{R}^n$ where $n = \dim E$, we may assume without loss of generality that $E = \mathbb{R}^n$.

We denote by $D^m(\mathbb{R}^n; F)$ the vector space of all functions $f : \mathbb{R}^n \to F$ which are of class $C^m (1 \leq m < \infty$) and have compact support. If $F = \mathbb{R}$, we write simply $D^m(\mathbb{R}^n)$. Also $D(\mathbb{R}^n; F)$ (resp. $D(\mathbb{R}^n)$) denotes the vector space of all functions $f : \mathbb{R}^n \to F$ (resp. $f : \mathbb{R}^n \to \mathbb{R}$) which are of class $C^\infty$ and have compact support.

In Schwartz [6] it was shown that the space $D(\mathbb{R}^n) \otimes F$ is dense in $D^m(\mathbb{R}^n; F)$ in the inductive limit topology. Since this topology is stronger than $\tau_u$, it follows that $D(\mathbb{R}^n) \otimes F$ is $\tau_u$-dense in $D^m(\mathbb{R}^n; F)$. Now, it is easily seen that $D^m(\mathbb{R}^n; F)$ is $\tau_u$-dense in $C^m(\mathbb{R}^n; F)$. Indeed, given $f \in C^m(\mathbb{R}^n; F)$, $K \subset \mathbb{R}^n$ compact and $\varepsilon > 0$, let $L \subset \mathbb{R}^n$ be a compact neighborhood of $K$. Let
Let \( \varphi \in D (\mathbb{R}^n) \) be such that \( \varphi (x) = 1 \) for all \( x \in L \). Then \( g = \varphi f \in D^m (\mathbb{R}^n ; F) \) and \( g (x) = f (x) \) for all \( x \in L \). Hence

\[
\| D^k g (x) - D^k f (x) \| = o < \varepsilon
\]

for all \( x \in K \) and \( o \leq k \leq m \). We have therefore proved the following

**Lemma 2.1.** \( C^m (\mathbb{R}^n) \otimes F \) is \( \tau_u \)-dense in \( C^m (\mathbb{R}^n ; F) \).

**Theorem 2.2.** Let \( E \) be a real, finite dimensional Banach space and let \( F \) be any real Banach space. Let \( A \in C^m (E ; F) \) be a polynomial algebra. Then \( A \) is \( \tau_u \)-dense in \( C^m (E ; F) \) if, and only if, the following conditions are satisfied:

1. for every \( x \in E \), there exists an \( f \in A \) such that \( f (x) = 0 \);
2. for every pair \( x, y \in E \) with \( x = y \), there exists an \( f \in A \) such that \( f (x) = f (y) \);
3. for every \( x \in E \) and for every \( v \in E \), with \( v = 0 \), there exists an \( f \in A \) such that \( Df (x) v = 0 \).

**Proof.** The necessity of the conditions is easily verified. Conversely, assume that \( A \) is a polynomial algebra satisfying conditions (1)-(3).

Let \( M = \{ u^* (f) ; u^* \in F^* , f \in A \} \). By Lemma 2.2. of [5], \( M \) is a subalgebra of \( C^m (E) \) such that \( M \otimes F \subseteq A \). The conditions (1)-(3) on \( A \) imply that \( M \) satisfies the hypothesis of Nachbin's theorem (see [4], p. 1550). Hence \( M \) is dense in \( C^m (E) \) in the \( \tau_u \) topology. It follows immediately that \( M \otimes F \) is \( \tau_u \)-dense in \( C^m (E) \otimes F \). By Lemma 2.1, \( M \otimes F \) is then \( \tau_u \)-dense in \( C^m (E ; F) \). Since \( M \otimes F \) is contained in \( A \), this completes the proof.

§ 3. INFINITE-DIMENSIONAL CASE

In this section \( E \) is a real, separable Hilbert space. We say that a subset \( A \subset C^m (E ; F) , m \geq 1 \), satisfies conditions \( (N_0) \) (see Lesmes [3]) if:

1. for every \( x \in E \), there exists an \( f \in A \) such that \( f (x) = 0 \);
2. for every pair \( x, y \in E \) with \( x \neq y \), there exists an \( f \in A \) such that \( f (x) = f (y) \);
3. for every \( x \in E \) and for every \( v \in E \), with \( v = 0 \), there exists an \( f \in A \) such that \( Df (x) v = 0 \);
4. there exists an orthonormal basis \( B = \{ e_n ; n \in \mathbb{N} \} \) of \( E \) and an integer \( M \in \mathbb{N} \) such that \( f \circ P_n \) belongs to \( A \), for every \( f \in A \) and \( n \geq M \), where \( P_n \) denotes the orthogonal projection of \( E \) onto \( E_n = \text{span} \{ e_1 , \ldots , e_n \} \).

For each \( n \in \mathbb{N} \), let \( j_n : E_n \to E \) be the linear isometry defined by \( j_n (x) = x \in E \), for all \( x \in E_n \). If \( f \in C^m (E ; F) \), we set \( T_n (f) = f \circ j_n \). Clearly, \( T_n \) is a continuous linear map from \( C^m (E ; F) \) into \( C^m (E_n ; F) \), when each space has its \( \tau_u \) topology.

**Lemma 3.1.** Let \( A \subset C^m (E ; F) \) be a polynomial algebra satisfying conditions (1)-(3) of \( (N_0) \). Then \( T_n (A) \) is \( \tau_u \)-dense in \( C^m (E_n ; F) \), for each \( n \in \mathbb{N} \).
Proof. Let \( n \in \mathbb{N} \). Since \( T_n \) is linear, \( T_n(A) \) is a vector subspace of \( \mathcal{C}^m(E_n; F) \). Let \( g \in T_n(A) \) and \( p \in \mathcal{P}_r(F; F), r \geq 1 \), be given. Let \( g = T_n(f), f \in A \). Then \( \rho \circ g = (\rho \circ f) \circ j_n = T_n(p \circ f) \). Since \( p \circ f \in A \), we conclude that \( \rho \circ g \) belongs to \( T_n(A) \), i.e., \( T_n(A) \) is a polynomial algebra.

Let \( x \in E_n \). Then \( j_n(x) \in E \), and by condition (1) of (N0), there exists \( \gamma \in A \) such that \( f(j_n(x)) = T_n(f(x)) = \gamma \). Let now \( x, y \in E_n \), with \( x \neq y \). Since \( j_n \) is one-to-one, \( j_n(x) \neq j_n(y) \). By condition (2) of (N0), there exists \( \gamma \in A \) such that \( f(j_n(x)) = f(j_n(y)), \) i.e. \( (T_n(f))(x) = (T_n(f))(y) \). Finally let \( x, v \in E_n \), with \( v \neq 0 \). Then \( j_n(v) \in E \) and \( j_n(v) \neq 0 \). By condition (3) of (N0), there exists \( \gamma \in A \) such that \( (Df(j_n(x)))(\gamma)(v) = 0 \). Hence \( T_n(A) \) satisfies the conditions of Theorem 2.2., and thus \( T_n(A) \) is \( \tau_\gamma \)-dense in \( \mathcal{C}^m(E_n; F) \), for each \( n \in \mathbb{N} \).

**Lemma 3.2.** Let \( f \in \mathcal{C}^m(E; F) \). Then the sequence \( \{ f \circ P_n \} \) converges to \( f \) in the \( \tau_\gamma \) topology.

**Proof.** Let \( K \) and \( L \) be two compact subsets of \( E \) and let \( \epsilon > 0 \) be given. Let \( r = \sup \{ \| h \| ; h \in L \} \).

Since \( D^k f \) is continuous \( (0 \leq k \leq m) \) and \( K \) is compact, we can find a real number \( \delta > 0 \) such that

\[
(1) \quad x \in K, y \in E, \| x - y \| < \delta \Rightarrow \| f(x) - f(y) \| < \epsilon \quad \text{and} \quad \| D^k f(x) - D^k f(y) \| < \epsilon/2(r + 1)^k, \quad 1 \leq k \leq m.
\]

Choose \( \eta > 0 \) such that \( 2sk(r + \eta)^{k-1} < \epsilon \), for all \( 1 \leq k \leq m \), where \( s = \max \{ \sup \{ \| D^k f(x) \| ; x \in \mathcal{K} \} ; 1 \leq k \leq m \} \). Since both \( K \) and \( L \) are compact, there exists an integer \( n_0 \in \mathbb{N} \) such that

\[
(2) \quad n \geq n_0, x \in K, h \in L \Rightarrow \| P_n x - x \| < \delta, \| P_n h - h \| < \eta.
\]

By (1) and (2) it follows that

\[
(3) \quad \| f(x) - f(P_n(x)) \| < \epsilon \quad \text{and} \quad \| D^k f(x) - D^k f(P_n(x)) \| < \epsilon/2 (r + 1)^k, \quad 1 \leq k \leq m,
\]

for all \( x \in K \) and \( n \geq n_0 \). Since \( \| P_n h \| \leq \| h \| \) for all \( h \in E \), the second inequality in (2) implies

\[
(4) \quad \| [D^k f(x)]^n(P_n h) - [D^k f(P_n x)]^n(P_n h) \| < \epsilon/2, \quad 1 \leq k \leq m,
\]

for all \( x \in K, h \in L \), and \( n \geq n_0 \). On the other hand we have

\[
(5) \quad \| [D^k f(x)]^n(P_n h) - [D^k f(x)^n h] \| \leq \| D^k f(x) \| \cdot k(r + \eta)^{k-1} \eta < \epsilon/2
\]

for all \( x \in K, h \in L \) and \( n \geq n_0 \). Combining (4) and (5) we get

\[
(6) \quad \| [D^k f(x)^n h - [D^k f(P_n x)^n(P_n h) \| < \epsilon, \quad 1 \leq k \leq m,
\]

for all \( x \in K, h \in L \) and \( n \geq n_0 \). Since \([D^k f(P_n x)^n(P_n h) = [D^k f(P_n x)^n(P_n h) \) for all \( x, h \in E \) and \( 1 \leq k \leq m \), (6) and the first inequality of (3) imply together \( P_{K,L}(f - f \circ P_n) < \epsilon \), for all \( n \geq n_0 \), which completes the proof.
Theorem 3.3. Let $A \subseteq C^m(E;F)$ be a polynomial algebra satisfying conditions (N$_0$). Then $A$ is $\tau_r$-dense in $C^m(E;F)$.

Proof. Let $f \in C^m(E;F)$, $K$ and $L$ compact subsets of $E$ and $\varepsilon > 0$ be given. By Lemma 3.2., there exists $n_1 \in \mathbb{N}$ such that $P_{K,L}(f - f \circ P_n) < \varepsilon/2$, for all $n \geq n_1$. Fix $n \in \mathbb{N}$ with $n > \max \{n_1, M\}$. Since $P_n$ is continuous and $K$ is compact, $K_n = P_n(K)$ is a compact subset of $E^m$. Choose $\delta > 0$ such that $\delta < \varepsilon/2(r + 1)^k$ for all $1 \leq k \leq m$, where $r = \sup \{\|P_nh\| : h \in L\}$. By Lemma 3.1., there exists $g \in A$ such that $P_{K_n}(T_n g - T_n(f \circ P_n)) < \delta$. Since $f_n \circ P_n = P_n$ and $P_n \circ f_n \circ P_n = P_n$, it follows that

\[ (1) \quad \|g(P_n(x)) - f(P_n(x))\| < \varepsilon/2 \]

and

\[ (2) \quad \|D^k(T_n g)(P_n x) - D^k(T_n(f \circ P_n))(P_n x)\| < \delta \]

for all $x \in K$, and $1 \leq k \leq m$. However, $[D^k(T_n g)(P_n x)](P_n h) = [D^k g(P_n x)](P_n h)$ and $[D^k(T_n(f \circ P_n))(P_n x)](P_n h) = [D^k (f \circ P_n)(P_n x)](P_n h)$ for all $x, h \in E$ and $1 \leq k \leq m$. Hence (2) implies

\[ (3) \quad \|D^k[g(P_n x)](P_n h) - [D^k(f \circ P_n)(P_n x)](P_n h)\| < \varepsilon/2 \]

for all $x \in K$, $h \in L$ and $1 \leq k \leq m$. Finally, (1) and (4) together imply $P_{K,L}(f - g \circ P_n) < \varepsilon/2$. Hence, $P_{K,L}(f - g \circ P_n) < \varepsilon$, and since $g \circ P_n$ belongs to $A$, $f$ belongs to the $\tau_r$-closure of $A$.

Corollary 3.4. The polynomial algebra $P_f(E;F)$ is dense in $C^m(E;F)$ in the $\tau_r$-topology.

Theorem 3.5. Let $\Phi : E \to F$ be a $C^1$-isomorphism of $E$ onto some open subset of $F$. Then $\{\rho \circ \Phi ; \rho \in P_f(F;F)\}$ is $\tau_r$-dense in $C^1(E;F)$ and $\{\rho \circ \Phi ; \rho \in P_f(F;R)\}$ is $\tau_r$-dense in $C^1(E)$, if $F$ is also a Hilbert space.

Proof. Let $g \in C^1(E;F), K$ and $L$ compact subsets of $E$, and $\varepsilon > 0$ be given. Let $A = \{\rho \circ \Phi ; \rho \in P_f(F;F)\}$.

Define $h : \Phi(E) \to F$ by $h(y) = g(\Phi^{-1}(y))$ for all $y \in \Phi(E)$. Then $h \in C^1(\Phi(E))$ and $g(x) = h(\Phi(x))$ for all $x \in E$. By Corollary 3.4. there exists $\rho \in P_f(F;F)$ such that

\[ (1) \quad \|\rho(y) - h(y)\| < \varepsilon, \quad \|D\rho(y)w - Dh(y)w\| < \varepsilon \]

for all $y \in \Phi(K)$ and $w \in [D\Phi(K)](L)$.

Since $D(h \circ \Phi)(x) = Dh(\Phi(x)) \circ D\Phi(x), D(\rho \circ \Phi)(x) = D\rho(\Phi(x)) \circ D\Phi(x)$ for $x \in E$, (1) implies

\[ (2) \quad \|\rho(\Phi(x)) - h(\Phi(x))\| < \varepsilon, \quad \|D(\rho \circ \Phi)(x) v - D(h \circ \Phi)(x) v\| < \varepsilon \]

for all $x \in K$ and $v \in L$. However $g = h \circ \Phi$ and $\rho \circ \Phi \in A$. Hence $A$ is $\tau_r$-dense in $C^1(E;F)$. 
The proof that \( \{ p \circ \Phi ; p \in \mathcal{P}_f(F; \mathbb{R}) \} \) is \( \tau_u \)-dense in \( \mathcal{C}^1(E) \) is analogous, and makes use of the fact that \( \mathcal{P}_f(F, \mathbb{R}) \) is \( \tau_u \)-dense in the algebra \( \mathcal{C}^1(F) \). This last fact is a corollary of Lesmes' theorem (see [3]).

**Example 3.6.** Let \( E \) be a real separable Hilbert space and let \( \Phi : E \to E \) be the map defined by
\[
\Phi(x) = x (1 + \|x\|^2)^{-1/2}
\]
for all \( x \in E \). Then \( \Phi \) is a \( C^\infty \)-isomorphism of \( E \) onto the open unit ball of \( E \).

**Bibliography**


