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On Polynomial Algebras of Continuously Differentiable Functions

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On Polynomial Algebras of Continuously Differentiable Functions.* Nota di JOÃO B. PROLLA, presentata (*) dal Corrisp. G. FICHERA.

RIASSUNTO. — Sia E uno spazio di Hilbert reale e separabile e sia F uno spazio di Banach reale. Viene esteso il teorema di Nachbin sulla densità delle algebre di funzioni di classe C^m a certe algebre polinomiali di funzioni da E ad F.

§ 1. INTRODUCTION

Let E and F be two real Banach spaces, with $F \neq \{0\}$. Then $C^m(E; F)$ denotes the vector space of all maps $f: E \to F$ which are of class C^m. We shall introduce two topologies on $C^m(E; F)$. The first one is the topology τ_u of uniform convergence of the functions and their derivatives on the compact subsets of E. It may be defined by the family of seminorms of the form

$$p_K(f) = \max\{\sup\{\|D^k f(x)\| ; x \in K\} ; 0 \leq k \leq m\}$$

where K is a compact subset of E. The second topology, denoted by τ_c, is defined by the family of seminorms of the form

$$p_{K,L}(f) = \max\{\sup\{\|[D^k f(x)]^\wedge(v)\| ; x \in K, v \in L\} ; 0 \leq k \leq m\},$$

where K and L are compact subsets of E and $T^\wedge(v) = T(v, \cdots, v)$, when $T \in \mathcal{L}_s({}^k E; F)$, $0 \leq k \leq m$.

If $F = \mathbf{R}$, then $C^m(E; F)$ is an algebra, denoted simply by $C^m(E)$. If $E = \mathbf{R}^n$ and $F = \mathbf{R}$, Nachbin proved in [4] an analogue of the Stone-Weierstrass theorem for the topology τ_u. In fact, he gave necessary and sufficient conditions for a subalgebra of $C^m(V)$ to be τ_u-dense, where V is an n-dimen-

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sional C^m-manifold, $m \geq 1$. If E is a real, separable Hilbert space and $F = \mathbf{R}$, J. Lesmes gave in [3] sufficient conditions for a subalgebra of $C^1(E)$ to be τ_u-dense.

In the case of a general F the space $C^m(E; F)$ is not an algebra. However, we can still get a Stone-Weierstrass theorem for the so called polynomial algebras. For each integer $n \geq 1$, $P_f(^nE; F)$ denotes the vector subspace of $C(E; F)$ generated by the set of all maps of the form $x \mapsto u^*(x)^n u$, where $u^* \in E^*$, the topological dual of E, and $u \in F$. The elements of $P_f(^nE; F)$ are called n-homogenuous continuous polynomials of finite type from E into F. The vector subspace generated by the union of all $P_f(^nE; F)$, $n \geq 1$, and the constant maps, is denoted by $P_f(E; F)$. A vector subspace $A \subset C(E; F)$ is called a *polynomial algebra* if, given $g \in A$ and $p \in P_f(^nF; F)$, where $n \geq 1$, then $p \circ g$ belongs to A. In our joint work [5] with S. Machado we proved that the Stone-Weierstrass theorem is true for polynomial algebras of continuous functions.

In this paper, we first extend Nachbin's theorem for polynomial algebras in the case in which E is finite dimensional, F is any real Banach space, and $C^m(E; F)$ has the τ_u topology. Afterwards, we apply this result to the case in which E is a real, separable Hilbert space, to obtain sufficient conditions for a polynomial algebra to be dense in $C^m(E; F)$ with the τ_c topology. As an application we show that, if $\Phi : E \to F$ is a C^1-isomorphism of E onto some open subset (e.g. the open unit ball) of F, then the F-valued polynomials in Φ are τ_c-dense in $C^1(E; F)$, and real valued polynomials in Φ are τ_u-dense in $C^1(E)$, when F is another real separable Hilbert space.

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§ 2. Vector-valued functions of n variables

In this section E is a finite-dimensional real Banach space and F is any real Banach space, not reduced to $\{0\}$. Since E is C^∞-isomorphic with $\mathbf{R}^n$ where $n = \dim E$, we may assume without loss of generality that $E = \mathbf{R}^n$.

We denote by $D^m(\mathbf{R}^n; F)$ the vector space of all functions $f : \mathbf{R}^n \to F$ which are of class C^m $(1 \leq m < \infty)$ and have compact support. If $F = \mathbf{R}$, we write simply $D^m(\mathbf{R}^n)$. Also $D(\mathbf{R}^n; F)$ (resp. $D(\mathbf{R}^n)$) denotes the vector space of all functions $f : \mathbf{R}^n \to F$ (resp. $f : \mathbf{R}^n \to \mathbf{R}$) which are of class C^∞ and have compact support.

In Schwartz [6] it was shown that the space $D(\mathbf{R}^n) \otimes F$ is dense in $D^m(\mathbf{R}^n; F)$ in the inductive limit topology. Since this topology is stronger than τ_u, it follows that $D(\mathbf{R}^n) \otimes F$ is τ_u-dense in $D^m(\mathbf{R}^n; F)$. Now, it is easily seen that $D^m(\mathbf{R}^n; F)$ is τ_u-dense in $C^m(\mathbf{R}^n; F)$. Indeed, given $f \in C^m(\mathbf{R}^n; F)$, $K \subset \mathbf{R}^m$ compact and $\varepsilon > 0$, let $L \subset \mathbf{R}^n$ be a compact neighborhood of K. Let

$\varphi \in D(\mathbf{R}^n)$ be such that $\varphi(x) = 1$ for all $x \in L$. Then $g = \varphi f \in D^m(\mathbf{R}^n; F)$ and $g(x) = f(x)$ for all $x \in L$. Hence

$$\| D^k g(x) - D^k f(x) \| = 0 < \varepsilon$$

for all $x \in K$ and $0 \leq k \leq m$. We have therefore proved the following

LEMMA 2.1. *$C^m(\mathbf{R}^n) \otimes F$ is τ_u-dense in $C^m(\mathbf{R}^n; F)$.*

THEOREM 2.2. *Let E be a real, finite dimensional Banach space and let F be any real Banach space. Let $A \in C^m(E; F)$ be a polynomial algebra. Then A is τ_u-dense in $C^m(E; F)$ if, any only if, the following conditions are satisfied:*

(1) *for every $x \in E$, there exists an $f \in A$ such that $f(x) \neq 0$;*

(2) *for every pair $x, y \in E$ with $x \neq y$, there exists an $f \in A$ such that $f(x) \neq f(y)$;*

(3) *for every $x \in E$, and for every $v \in E$, with $v \neq 0$, there exists an $f \in A$ such that $Df(x)v \neq 0$.*

Proof. The necessity of the conditions is easily verified. Conversely, assume that A is a polynomial algebra satisfying conditions (1)–(3).

Let $M = \{u^*(f); u^* \in F^*, f \in A\}$. By Lemma 2.2. of [5], M is a subalgebra of $C^m(E)$ such that $M \otimes F \subset A$. The conditions (1)–(3) on A imply that M satisfies the hypothesis of Nachbin's theorem (see [4], p. 1550). Hence M is dense in $C^m(E)$ in the τ_u topology. It follows immediately that $M \otimes F$ is τ_u-dense in $C^m(E) \otimes F$. By Lemma 2.1, $M \otimes F$ is then τ_u-dense in $C^m(E; F)$. Since $M \otimes F$ is contained in A, this completes the proof.

§ 3. INFINITE-DIMENSIONAL CASE

In this section E is a real, separable Hilbert space. We say that a subset $A \subset C^m(E; F)$, $m \geq 1$, satisfies conditions (N_0) (see Lesmes [3]) if:

(1) for every $x \in E$, there exists an $f \in A$ such that $f(x) \neq 0$;

(2) for every pair $x, y \in E$, with $x \neq y$, there exists an $f \in A$ such that $f(x) \neq f(y)$;

(3) for every $x \in E$ and for every $v \in E$, with $v \neq 0$, there exists an $f \in A$ such that $Df(x)v \neq 0$;

(4) there exists an orthonormal basis $B = \{e_n; n \in \mathbf{N}\}$ of E and an integer $M \in \mathbf{N}$ such that $f \circ P_n$ belongs to A, for every $f \in A$ and $n \geq M$, where P_n denotes the orthogonal projection of E onto $E_n = \text{span}\{e_1, \cdots, e_n\}$.

For each $n \in \mathbf{N}$, let $j_n : E_n \to E$ be the linear isometry defined by $j_n(x) = x \in E$, for all $x \in E_n$. If $f \in C^m(E; F)$, we set $T_n(f) = f \circ j_n$. Clearly, T_n is a continuous linear map from $C^m(E; F)$ into $C^m(E_n; F)$, when each space has its τ_u topology,

LEMMA 3.1. *Let $A \subset C^m(E; F)$ be a polynomial algebra satisfying conditions (1)–(3) of (N_0). Then $T_n(A)$ is τ_u-dense in $C^m(E_n; F)$, for each $n \in \mathbf{N}$.*

Proof. Let $n \in \mathbf{N}$. Since T_n is linear, $T_n(A)$ is a vector subspace of $C^m(E_n ; F)$. Let $g \in T_n(A)$ and $p \in P_f({}^rF ; F)$, $r \geq 1$, be given. Let $g = T_n(f)$, $f \in A$. Then $p \circ g = p \circ (f \circ j_n) = (p \circ f) \circ j_n = T_n(p \circ f)$. Since $p \circ f \in A$, we conclude that $p \circ g$ belongs to $T_n(A)$, i.e., $T_n(A)$ is a polynomial algebra.

Let $x \in E_n$. Then $j_n(x) \in E$, and by condition (1) of (N_0), there exists $f \in A$ such that $f(j_n(x)) = T_n f(x) \neq 0$. Let now $x, y \in E_n$, with $x \neq y$. Since j_n is one-to-one, $j_n(x) \neq j_n(y)$. By condition (2) of (N_0), there exists $f \in A$ such that $f(j_n(x)) \neq f(j_n(y))$, i.e. $(T_n f)(x) \neq (T_n f)(y)$. Finally let $x, v \in E_n$, with $v \neq 0$. Then $j_n(v) \in E$ and $j_n(v) \neq 0$. By condition (3) of (N_0), there exists $f \in A$ such that $[Df(j_n(x))] j_n(v) \neq 0$. Since j_n is linear, $D(T_n f)(x) = Df(j_n(x)) \circ j_n$ and therefore $[D(T_n f)(x)] v = [Df(j_n(x))] j_n(v) \neq 0$. Hence $T_n(A)$ satisfies the conditions of Theorem 2.2., and thus $T_n(A)$ is τ_u-dense in $C^m(E_n; F)$, for each $n \in \mathbf{N}$.

LEMMA 3.2. *Let $f \in C^m(E ; F)$. Then the sequence $\{f \circ P_n\}$ converges to f in the τ_c topology.*

Proof. Let K and L be two compact subsets of E and let $\varepsilon > 0$ be given. Let $r = \sup \{\|h\| ; h \in L\}$.

Since $D^k f$ is continuous $(0 \leq k \leq m)$ and K is compact, we can find a real number $\delta > 0$ such that

$$(1) \quad x \in K, y \in E, \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon \text{ and } \|D^k f(x) - D^k f(y)\| < \varepsilon/2(r+1)^k, \quad 1 \leq k \leq m.$$

Choose $\eta > 0$ such that $2sk(r+\eta)^{k-1} < \varepsilon$, for all $1 \leq k \leq m$, where $s = \max\{\sup\{\|D^k f(x)\| ; x \in K\} ; 1 \leq k \leq m\}$. Since both K and L are compact, there exists an integer $n_0 \in \mathbf{N}$ such that

$$(2) \quad n \geq n_0, x \in K, h \in L \Rightarrow \|P_n x - x\| < \delta, \|P_n h - h\| < \eta.$$

By (1) and (2) it follows that

$$(3) \quad \|f(x) - f(P_n(x))\| < \varepsilon \text{ and } \|D^k f(x) - D^k f(P_n(x))\| < \varepsilon/2(r+1)^k, \quad 1 \leq k \leq m,$$

for all $x \in K$ and $n \geq n_0$. Since $\|P_n h\| \leq \|h\|$ for all $h \in E$, the second inequality in (2) implies

$$(4) \quad \|[D^k f(x)]^\wedge(P_n h) - [D^k f(P_n x)]^\wedge(P_n h)\| < \varepsilon/2, \quad 1 \leq k \leq m,$$

for all $x \in K, h, \in L$, and $n \geq n_0$. On the other hand we have

$$(5) \quad \|[D^k f(x)]^\wedge(P_n h) - [D^k f(x)^\wedge h\| \leq \|D^k f(x)\| \cdot k(r+\eta)^{k-1} \eta < \varepsilon/2$$

for all $x \in K, h \in L$ and $n \geq n_0$. Combining (4) and (5) we get

$$(6) \quad \|[D^k f(x)]^\wedge h - [D^k f(P_n x)]^\wedge(P_n h)\| < \varepsilon, \quad 1 \leq k \leq m,$$

for all $x \in K, h \in L$ and $n \geq n_0$. Since $[D^k(f \circ P_n)(x)]^\wedge h = [D^k f(P_n x)]^\wedge(P_n h)$ for all $x, h \in E$ and $1 \leq k \leq m$, (6) and the first inequality of (3) imply together $P_{K,L}(f - f \circ P_n) < \varepsilon$, for all $n \geq n_0$, which completes the proof.

THEOREM 3.3. *Let* $A \subset C^m(E;F)$ *be a polynomial algebra satisfying conditions* (N_0). *Then* A *is* τ_c*-dense in* $C^m(E;F)$.

Proof. Let $f \in C^m(E;F)$, K and L compact subsets of E and $\varepsilon > 0$ be given. By Lemma 3.2., there exists $n_1 \in \mathbf{N}$ such that $P_{K,L}(f - f \circ P_n) < \varepsilon/2$, for all $n \geq n_1$. Fix $n \in \mathbf{N}$ with $n > \max(n_1, M)$. Since P_n is continuous and K is compact, $K_n = P_n(K)$ is a compact subset of E_n. Choose $\delta > 0$ such that $\delta < \varepsilon/2\,(r+1))^k$ for all $1 \leq k \leq m$, where $r = \sup\{\|P_n h\| ; h \in L\}$. By Lemma 3.1., there exists $g \in A$ such that $P_{K_n}(T_n g - T_n(f \circ P_n)) < \delta$. Since $j_n \circ P_n = P_n$ and $P_n \circ j_n \circ P_n = P_n$, it follows that

(1) $\|g(P_n(x)) - f(P_n(x))\| < \varepsilon/2$, and

(2) $\|[D^k(T_n g)](P_n x) - [D^k(T_n(f \circ P_n))](P_n x)\| < \delta$

for all $x \in K$, and $1 \leq k \leq m$. However, $[[D^k(T_n g)](P_n x)]^\wedge h = [D^k g(P_n x)]^\wedge (P_n h)$ and $[[D^k(T_n(f \circ P_n))](P_n x)]^\wedge h = [[D^k(f \circ P_n)](P_n x)]^\wedge (P_n h)$ for all $x, h \in E$ and $1 \leq k \leq m$. Hence (2) implies

(3) $\|[D^k g(P_n x)]^\wedge (P_n h) - [[D^k(f \circ P_n)](P_n x)]^\wedge (P_n h)\| < \varepsilon/2$

for all $x \in K$, $h \in L$ and $1 \leq k \leq m$. On the other hand, $[D^k(g \circ P_n)(x)]^\wedge h = [D^k g(P_n x)]^\wedge (P_n h)$ and $[D^k(f \circ P_n)(x)]^\wedge h = [D^k(f \circ P_n^2)(x)]^\wedge h = [D^k(f \circ P_n) \cdot (P_n x)]^\wedge (P_n h)$. Hence, (3) implies

(4) $\|[D^k(g \circ P_n)(x)]^\wedge h - [D^k(f \circ P_n)(x)]^\wedge h\| < \varepsilon/2$

for all $x \in K$, $h \in L$ and $1 \leq k \leq m$. Finally, (1) and (4) together imply $P_{K,L}(f \circ P_n - g \circ P_n) < \varepsilon/2$. Hence, $P_{K,L}(f - g \circ P_n) < \varepsilon$, and since $g \circ P_n$ belongs to A, f belongs to the τ_c-closure of A.

COROLLARY 3.4. *The polynomial algebra* $P_f(E;F)$ *is dense in* $C^m(E;F)$ *in the* τ_c *topology.*

THEOREM 3.5. *Let* $\Phi : E \to F$ *be a* C^1*-isomorphism of* E *onto some open subset of* F. *Then* $\{p \circ \Phi ; p \in P_f(F;F)\}$ *is* τ_c*-dense in* $C^1(E;F)$ *and* $\{p \circ \Phi ; p \in P_f(F;\mathbf{R})\}$ *is* τ_u*-dense in* $C^1(E)$, *if* F *is also a Hilbert space.*

Proof. Let $g \in C^1(E;F)$, K and L compact subsets of E, and $\varepsilon > 0$ be given. Let $A = \{p \circ \Phi ; p \in P_f(F;F)\}$.

Define $h : \Phi(E) \to F$ by $h(y) = g(\Phi^{-1}(y))$ for all $y \in \Phi(E)$. Then $h \in C^1(\Phi(E))$ and $g(x) = h(\Phi(x))$ for all $x \in E$. By Corollary 3.4. there exists $p \in P_f(F;F)$ such that

(1) $\|p(y) - h(y)\| < \varepsilon$, $\|Dp(y)w - Dh(y)w\| < \varepsilon$

for all $y \in \Phi(K)$ and $w \in [D\Phi(K)](L)$.

Since $D(h \circ \Phi)(x) = Dh(\Phi(x)) \circ D\Phi(x)$, $D(p \circ \Phi)(x) = Dp(\Phi(x)) \circ D\Phi(x)$ for $x \in E$, (1) implies

(2) $\|p(\Phi(x)) - h(\Phi(x))\| < \varepsilon$, $\|D(p \circ \Phi)(x)v - D(h \circ \Phi)(x)v\| < \varepsilon$

for all $x \in K$ and $v \in L$. However $g = h \circ \Phi$ and $p \circ \Phi \in A$. Hence A is τ_c-dense in $C^1(E;F)$.

The proof that $\{ p \circ \Phi ; p \in P_f(F ; \mathbf{R})\}$ is τ_u-dense in $C^1(E)$ is analogous, and makes use of the fact that $P_f(F ; \mathbf{R})$ is τ_u-dense in the algebra $C^1(F)$. This last fact is a corollary of Lesmes' theorem (see [3]).

Example 3.6. Let E be a real separable Hilbert space and let $\Phi : E \to E$ be the map defined by

$$\Phi(x) = x (1 + \| x \|^2)^{-1/2}$$

for all $x \in E$. Then Φ is a C^∞-isomorphism of E onto the open unit ball of E.

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