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Vincenzo Malvestuto, Francesco Zirilli

Inertial waves in a homogeneous ocean. The dispersion relation for standing waves in presence of the horizontal component of Earth rotation

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Oceanografia. — *Inertial waves in a homogeneous ocean. The dispersion relation for standing waves in presence of the horizontal component of Earth rotation.* Nota di VINCENZO MALVESTUTO e FRANCESCO ZIRILLI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si studiano le equazioni di Eulero linearizzate del moto in un oceano non viscoso omogeneo ruotante. Si considerano, i seguenti due casi: *a*) l'asse di rotazione è verticale, *b*) l'angolo fra l'asse di rotazione e la verticale è non nullo ed uguale alla latitudine. Nella situazione *a*) si ottiene l'insieme delle frequenze caratteristiche per le onde inerziali; nella situazione *b*) si deriva un'interessante equazione per l'evoluzione della superfice e una relazione di dispersione per le « standing surface waves ».

In a preceding paper [1], dealing with the generation and the propagation of internal waves in a stratified inviscid ocean, we disregarded the inclination of Earth axis to the vertical and assumed throughout the development:

$$\mu = 0 ,$$

where μ is defined as twice the horizontal component of the angular velocity $\underline{\Omega}$ of Earth in its rotation:

$$\mu = 2\,\Omega \cos\theta \qquad (\theta \text{ is the latitude}).$$

The quantity μ is an important parameter, suitable to characterize the vertical effects of the Coriolis force upon the motion of the fluid. In this paper we intend to take certain account of such effects, after having obtained in the limit $\mu = 0$ and within the hydrostatic approximation the explicit characteristic frequencies of the allowed inertial waves.

In what follows the fluid is to be taken as homogeneous and incompressible; moreover we ignore, as usual, turbulence and viscosity.

If we assume that our ocean is undergoing little perturbations from an initial stable condition of equilibrium, so that its time-evolution may be regarded as a first order correction to the initial homogeneous conditions, the eulerian equations of the motion in the first order approximation are:

$$(1)\qquad \begin{aligned} &\text{(a)} \quad u_t = \lambda v - \mu w + \frac{1}{\rho}\, p_x \\ &\text{(b)} \quad v_t = - \lambda u + \frac{1}{\rho}\, p_y \\ &\text{(c)} \quad w_t = \phantom{\lambda v - {}} \mu u + \frac{1}{\rho}\, p_z \\ &\text{(d)} \quad u_x + v_y + w_z = 0 \end{aligned}$$

where the density ρ is a datum of the problem.

(*) Nella seduta del 14 novembre 1974.

Let us consider the first three equations (a), (b), and (c). Their structure is the following:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathrm{A} f(t) + b(t) \qquad f(t) \in \mathrm{H}$$

where H is a suitable Sobolev space, and we have set by definition:

$$f(t) \equiv \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{2}$$

a three component vector; moreover we set

$$\mathrm{A} \equiv \begin{pmatrix} 0 & \lambda & -\mu \\ -\lambda & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix} \tag{3}$$

a bounded operator on the H-space, and finally

$$b(t) \equiv -\frac{1}{\rho} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}. \tag{4}$$

Now the first step is the construction of the semigroup generated by the bounded operator A, exp $(t\mathrm{A})$. It is useful in view of this to introduce the quantity σ, which dimensionally is a frequency, defined by means of the equation:

$$\sigma^2 \equiv \lambda^2 + \mu^2 = 4\,\Omega^2. \tag{5}$$

In terms of this fundamental frequency it is easy to write the entire semigroup in explicit form:

$$e^{t\mathrm{A}} = \begin{pmatrix} \cos \sigma t & \frac{\lambda}{\sigma} \sin \sigma t & -\frac{\mu}{\sigma} \sin \sigma t \\ -\frac{\lambda}{\sigma} \sin \sigma t & 1 - \frac{\lambda^2}{\sigma^2}(1 - \cos \sigma t) & \frac{\lambda\mu}{\sigma^2}(1 - \cos \sigma t) \\ \frac{\mu}{\sigma} \sin \sigma t & \frac{\lambda\mu}{\sigma}(1 - \cos \sigma t) & 1 - \frac{\mu^2}{\sigma^2}(1 - \cos \sigma t) \end{pmatrix}. \tag{6}$$

By applying the generalized Duhamel principle, we get from eq. (1) (a), (b), and (c):

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -\frac{1}{\rho} \int_0^t \mathrm{d}s \, e^{(t-s)\mathrm{A}} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}_{x,y,z,s}. \tag{7}$$

We now derive under the integral sign with respect to space variables, substitute in the continuity equation (l.d) and, after making the Laplace transform with respect to the time variable, we obtain, as in ref. [1], the following equation:

$$(8) \qquad [\omega(\omega^2+\sigma^2)]^{-1}\{\omega^2 P_{xx}+(\omega^2+\mu^2)P_{yy}+(\omega^2+\lambda^2)P_{zz}+2\lambda\mu P_{yz}\}=0$$

where $P(x, y, z, \omega)$ is the Laplace transform of $p(x, y, z, t)$.

The boundary conditions for eq. (8) are, as in ref. [1]:

$$(9) \qquad \begin{aligned} &P(x, y, 0, \omega) = \rho g A(\omega)\sin Kx \\ &\frac{\partial P}{\partial z}(x, y, -h, \omega) = 0, \end{aligned}$$

having regarded our ocean as infinitely extended in x- and y-directions and having supposed the external forces to maintain fixed the shape of the ocean surface:

$$\zeta(x, y, \omega) = A(\omega)\,\text{sen}\, Kx,$$

namely the Laplace transform of the vertical displacement of free surface.

If $\mu = 0$, the solution of (8) and (9) is:

$$(10) \qquad P(x, z, \omega) = \rho g A(\omega)\sin Kx\cosh(\gamma z)[1+\text{tgh}(\gamma h)\,\text{tgh}(\gamma z)],$$

where

$$\gamma \equiv K\omega[\omega^2+\lambda^2]^{-\frac{1}{2}}.$$

The characteristic frequencies are the poles of (10) respect to ω. An easy computation shows that these poles are:

$$(11) \qquad \omega_l = \pm 2i\Omega\left[1+\left(\frac{Kh}{l\pi}\right)^2\right]^{-\frac{1}{2}}, \qquad l = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots,$$

use having been made of the identity: $\lambda = 2\Omega$, valid as $\mu = 0$.

A quite similar result, derived in a different way, is contained in a paper by Saint-Guily [2], where the A., instead of considering fixed the shape of sea surface, supposes the free surface to be forced in the vertical direction for a finite time interval by a sinusoidal perturbation, proportional to $\sin Kx$. It is straightforward to find the connection between such condition and our approach.

Now we turn back to eq. (8) in order to derive a surface evolution equation in $\zeta(x, y, \omega)$ (which is no longer considered as maintained fixed by external forces), using the hydrostatic approximation not only on the surface but in the whole fluid domain. This approximation is now possible, because of the homogeneity of the ocean, that is: $\rho =$ const.; we can therefore write:

$$P = \rho g\zeta(x, y, \omega),$$

and so P does not depend on z.

The equation (8) then becomes:

$$(8') \qquad [\omega(\omega^2+\sigma^2)]^{-1}[\omega^2 P_{xx}+(\omega^2+\mu^2)P_{yy}] = 0.$$

If we look for the propagation of surface standing waves described by:

$$P(x, y, \omega) = C \sin K_1 x \sin K_2 y \frac{\omega}{\omega^2 + \nu^2}, \tag{12}$$

it is easy to derive the following dispersion relation:

$$\omega^2 K_1^2 + (\omega^2 + \mu^2) K_2^2 = 0,$$

that is:

$$\omega = \pm i\mu \frac{K_2}{\sqrt{K_1^2 + K_2^2}}. \tag{13}$$

Eq. (13) makes clear the role played by μ in connecting the wave-numbers K_1 and K_2 with time periodicity, as soon as we recognize by (12) that the only poles or allowed frequencies are

$$\omega = \pm i\nu. \tag{14}$$

REFERENCES

[1] V. MALVESTUTO, E. SALUSTI and F. ZIRILLI – *On a semigroup approach to oceanography: the treatment of the linearized equations for a non viscous sea*, to appear in « Rendiconti Acc. Naz. dei Lincei ».

[2] B. SAINT-GUILY (1971) – *Sur les ondes d'inertie en milieu homogene*, « C. R. Acad. Sc. Paris », *222*, 2531–2532.