Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Antonio Fasano, Mario Primicerio

One phase and two-phase free boundary problems of general type for the heat equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.5, p. 387–390.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_5_387_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Fisica matematica. — *One phase and two-phase free boundary problems of general type for the heat equation* (*). Nota di ANTONIO FASANO e MARIO PRIMICERIO, presentata (**) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Vengono enunciati teoremi di esistenza ed unicità per le soluzioni di problemi a contorno libero per l'equazione unidimensionale del calore. I risultati riguardano ampie classi di problemi a una e a due fasi.

The purpose of this Note is to announce some significant results on the following general classes of free boundary problems for the heat equation in one space dimension:

I. ONE-PHASE PROBLEMS

(1) $$u_{xx} - u_t = q(x,t) \quad \text{in} \quad D_T = \{(x,t) : 0 < x < s(t),\ 0 < t < T\}$$
$$s(0) \equiv b \geq 0$$

(2) $$u(x,0) = h(x), \qquad 0 \leq x \leq b, \quad \text{if } b > 0$$

(3) $$u(s(t),t) = f(s(t),t), \qquad 0 < t < T$$

(4) $$u_x(s(t),t) = \lambda(s(t),t)\,\dot{s}(t) + \mu(s(t),t), \qquad 0 < t < T$$

and

(5) $$u(0,t) = \varphi(t), \qquad 0 < t < T$$

or

(5′) $$u_x(0,t) = g(u(0,t),t), \qquad 0 < t < T$$

where $q(x,t)$, $h(x)$, $f(x,t)$, $\lambda(x,t)$, $\mu(x,t)$, $\varphi(t)$, $g(\xi,t)$ are given functions and $|\lambda(x,t)| \geq \lambda_0 > 0$ in $Q = \{(x,t) : 0 \leq x, 0 \leq t\}$: problems with $\lambda \equiv 0$ (i.e. with Cauchy data on the free boundary) can also be reduced to scheme (1)-(5) under suitable assumptions (see [1], [2] for details).

II. TWO-PHASE PROBLEMS

(6) $$a_1 u_{xx}^{(1)} - u_t^{(1)} = q^{(1)}(x,t), \qquad 0 < x < s(t), \quad 0 < t < T$$

(6′) $$a_2 u_{xx}^{(2)} - u_t^{(2)} = q^{(2)}(x,t), \qquad s(t) < x < 1, \quad 0 < t < T$$
$$s(0) \equiv b \in [0, 1]$$

(*) Lavoro eseguito nell'ambito del Gruppo Nazionale per la Fisica Matematica del C.N.R.
(**) Nella seduta del 14 novembre 74.

(7) $u^{(1)}(x, 0) = h(x) \qquad 0 \leq x \leq b, \quad \text{if } b > 0$

(7′) $u^{(2)}(x, 0) = h(x) \qquad b \leq x \leq 1, \quad \text{if } b < 1$

(8) $u^{(i)}(s(t), t) = f(s(t), t), \qquad 0 < t < T, \qquad i = 1, 2$

(9) $\chi^{(1)}(s(t), t)\, u_x^{(1)}(s(t), t) + \chi^{(2)}(s(t), t)\, u_x^{(2)}(s(t), t) =$
$= \lambda(s(t), t)\, \dot{s}(t) + \mu(s(t), t), \qquad 0 < t < T$

(10) $u^{(1)}(0, t) = \varphi^{(1)}(t), \qquad 0 < t < T$

(10′) $u^{(2)}(1, t) = \varphi^{(2)}(t), \qquad 0 < t < T.$

One or both of the conditions (10) can be replaced by the corresponding one in

(11) $u_x^{(1)}(0, t) = g^{(1)}(u^{(1)}(0, t), t), \qquad 0 < t < T$

(11′) $u_x^{(2)}(1, t) = g^{(2)}(u^{(2)}(1, t), t), \qquad 0 < t < T.$

The constants $a_i > 0$ and the functions $q^{(i)}(x, t)$, $h(x)$, $f(x, t)$, $\lambda(x, t)$, $\chi^{(i)}(x, t)$, $\mu(x, t)$, $\varphi^{(i)}(t)$, $g^{(i)}(\xi, t)$ are given. In particular, $|\chi^{(i)}(x, t)| \leq X$, $|\lambda(x, t)| \geq \lambda_0$ for some positive constants X and λ_0 and $(x, t) \in Q$.

Here we will state existence and uniqueness theorems for classical solutions (defined in the usual sense) to problems (I) and (II). We shall also sketch some indications on the proofs. In a paper which is to appear soon the extended proofs will be given together with additional results: a summary of it was communicated at the International Congress of Mathematicians in Vancouver (August 1974).

Problems (I) and (II) are the most general formulation at our knowledge of Stefan type problems (see [3], [4] for references): indeed, no sign restrictions are imposed on the data and $f, \lambda, \mu, \nu, \chi^{(i)}$ are assumed to depend both on space and on time.

Now we state our main theorems in their simplest form (1):

THEOREM 1. *Let* $b > 0$. *If* $f_{xx} - f_t - q$ *is bounded and locally Hölder–continuous in* Q; *if* $f_x, \mu, \lambda, \lambda_x, \lambda_t$ *are continuous*, $\sup_Q |\mu + f_x| = M$; *if* φ *is continuous for* $t \geq 0$, *is continuous in* $[0, b]$ *and* $|h(x) - f(b, 0)| \leq H(b - x)$ *for some positive constant* H; *then for any* $\alpha \in (0, b)$ *and* $A > (M + 2H)/\lambda_0$, *a time interval* $(0, T_0)$ *can be determined in which a unique solution of problem* (1)–(5) *exists such that* $|\dot{s}(t)| \leq A$.

(1) We are stating only results concerning problems (1)–(5) and (6)–(10). We also proved that similar results hold when conditions (5′) and (11), (11′) are prescribed instead of (5) and (10), (10′) respectively. The proof is given under suitable assumptions on the continuity and on the asymptotical behavior of $g, g^{(1)}, g^{(2)}$.

The time interval $(0, T)$ *in which the solution can be actually constructed is shown to be either infinite or bounded by* $T = T_S$ *such that* $\lim_{t \to T_S -} s(t) = 0$ *and/or* $\limsup_{t \to T_S -} |\dot{s}(t)| = +\infty$.

THEOREM 2. *Let* $0 < b < 1$. *Assume that the functions* $f_{xx} - f_t - q^{(i)}$ $i = 1, 2$ *are bounded and locally Hölder continuous in* Q *and that the functions* f, λ, μ, h *satisfy the same hypotheses listed in Theorem 1. Moreover, suppose that* $\varphi^{(i)}(t)$, $i = 1, 2$ *are continuous for* $t \geq 0$ *and that* $\chi^{(i)}$, $\chi_x^{(i)}$, $\chi_{xx}^{(i)}$, $\chi_t^{(i)}$ *are continuous in* Q. *Then for any* $A > 2X(M + 2H)/\lambda_0$, *a time interval* $(0, T_0)$ *can be determined in which a unique solution of problem* (6)–(10′) *exists such that* $|\dot{s}(t)| \leq A$.

Quite similar remarks to the ones made in Theorem 1 on the continuation of the solution over a larger time interval are also valid in this case.

We shall now outline the main steps of the proof of Theorem 1.

First, remark that there is no loss of generality in assuming $f \equiv 0$ in (3): re-define $u = u - f$ and change the data accordingly.

Next, define a sequence of approximate solutions *via* the recursive relationship $(k = 0, 1, 2 \cdots)$:

$$\lambda(s_k(t), t)\, \dot{s}_{k+1}(t) = u_{k,x}(s_k(t), t) + \mu(s_k(t), t) \tag{12}$$

where $u_k(x, t)$ solves problem (1), (2), (3), (5) with s replaced by s_k and $s_0(t)$ can be chosen arbitrarily in a suitable class (e.g. $s_0(t) \equiv b$).

The crucial point of the proof is to determine a uniform lower estimate of the time interval $(0, T_0)$ in which $s_k(t) > 0$ and $|\dot{s}_k(t)| \leq A$, $k = 0, 1, 2 \cdots$.

Then it is shown that subsequences $\{s_{k'}\}, \{u_{k'}\}$ are respectively convergent to a Lipschitz-continuous function $s(t)$ and to the corresponding solution of (1), (2), (3) and (5).

The final step of the existence proof is to demonstrate that the pair (s, u) satisfies the free boundary condition (4). This is achieved through the following integral reformulation of (4):

$$\begin{aligned} \frac{1}{2}\Lambda^2(s(t), t) - \frac{1}{2}\Lambda^2(b, 0) = \int_{D_t} \{ \Lambda(x, \tau)\, q(x, \tau) - \\ - u(x, \tau)\,[\lambda_x(x, \tau) + \Lambda_\tau(x, \tau)\} \, dx\, d\tau + \int_0^{s(t)} \Lambda(x, t)\, u(x, t)\, dx + \\ + \int_0^t \Lambda(s(\tau), \tau)\, \{\Lambda_\tau(s(\tau), \tau) - \mu(s(\tau), \tau)\}\, d\tau - \\ - \int_0^b \Lambda(x, 0)\, h(x)\, dx - \int_0^t \lambda(0, \tau)\, \varphi(\tau)\, d\tau, \end{aligned} \tag{13}$$

where $\Lambda(x, t) = \int_0^x \lambda(y, t)\, dy$.

Uniqueness is a consequence of a theorem of continuous dependence of the solution on the data, which we have established using tools similar to the ones of [5], [6].

The final statement of Theorem 1 (which extends the conclusions of [9]) is achieved by iteration.

It is worth noting that the scheme (12) is a generalization of the method used by Evans [7] and by Sestini [8].

The proof of Theorem 2 is basically grounded on the same ideas. It appears to be more direct and more effective (also in view of the generality of problems to which it is applicable) than the approach announced in [10] for the usual two-phase Stefan problem.

The case $b = 0$ requires a different and very delicate analysis involving the a-priori study of the asymptotical behavior of $s(t)$ near $t = 0$. Analogous results are obtained under additional assumptions on the data, in order to ascertain the monotonicity of $s(t)$ in a neighborhoud of $t = 0$.

A similar technique allows to release the assumption on the Lipschitz-continuity of the initial datum $h(x)$ at $x = b$.

References

[1] A. Schatz (1969) – *Free boundary problems of Stefan type with prescribed flux*, « J. Math. Anal. Appl. », *28*, 569–579.

[2] A. Fasano – *Alcune osservazioni su una classe di problemi a contorno libero per l'equazione del calore*. To appear on « Boll. U.M.I. ».

[3] L. I. Rubinstein (1971) – *The Stefan Problem*, « A.M.S. Translations of Mathematical Monographs », *27*, Providence, R.I.

[4] M. Primicerio – *Sui problemi a contorno libero per l'equazione del calore*. To appear on « Rend. Sem. Mat. Univ. », Torino.

[5] J. R. Cannon and J. Douglas (1967) – *The stability of the boundary in a Stefan problem*, « Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat. », *21*, 83–91.

[6] A. Fasano and M. Primicerio (1972) – *Il problema di Stefan con condizioni al contorno non lineari*, « Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat. », *26*, 711–737.

[7] G.W. Evans II (1971) – *A note on the existence of a solution to a problem of Stefan*, « Quart. Appl. Math. », *9*, 185–193.

[8] G. Sestini (1952) – *Esistenza di una soluzione in problemi analoghi a quello di Stefan*, « Riv. Mat. Univ. Parma », *3*, 3–22.

[9] B. Sherman (1971) – *General one-phase Stefan problems and free boundary problems for the heat equation with Cauchy data prescribed on the free boundary*, « SIAM J. Appl. Math. », *20*, 555–570.

[10] J. R. Cannon, D. B. Henry and D. B. Kotlow (1974) – *Continuous differentiability of the free boundary for weak solutions of the Stefan problem*, « Bull. A.M.S. », *80*, 45–84.