Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

Andrea Donato

## On supplementary conservation laws for second order hyperbolic conservative equation

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **57** (1974), n.5, p. 380–386.

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<http://www.bdim.eu/item?id=RLINA_1974_8_57_5_380_0>

**Fisica matematica.** — *On supplementary conservation laws for second order hyperbolic conservative equation* (*). Nota di ANDREA DONATO (**), presentata (***) dal Socio D. GRAFFI.

RIASSUNTO. — In questa Nota si determina la classe delle equazioni conservative del secondo ordine di forma canonica che ammettono una equazione di conservazione supplementare, seguendo un procedimento dovuto a G. Boillat. Si ha occasione di illustrare il metodo su un esempio particolare.

In a recent paper [1] G. Boillat found a necessary and sufficient condition for a conservative hyperbolic system of first order to have a supplementary conservation law.

In this paper, using the procedure established in [1], we determine the class of conservation equations of second order in canonical form admitting a supplementary conservation equation. We then give the explicit form of this conservation equation and show the usefulness of the method in the case of a non linear string equation [2]. In the last part we study the problem of obtaining directly the conservation equation and we find the same results.

1. Let us consider the following hyperbolic conservative system of N equations (1)

$$\partial_t \boldsymbol{u} + \partial_i \boldsymbol{f}^i(\boldsymbol{u}) = 0 \qquad (i = 1, 2, \cdots, m) \tag{1}$$

which can be written

$$\boldsymbol{u}_t + A^i(\boldsymbol{u})\,\boldsymbol{u}_i = 0 \quad ; \quad A^i = \nabla \boldsymbol{f}^i\,,\ \boldsymbol{u}_t = \partial_t \boldsymbol{u}\,,\ \boldsymbol{u}_i = \partial_i \boldsymbol{u}\,, \tag{1'}$$

with $\nabla = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \cdots, \frac{\partial}{\partial u_N}\right)$. The system (1) is hyperbolic so that the matrix $A_n = A^i n_i \left(\boldsymbol{n}^2 = \sum_i n_i^2 = 1\right)$ has N eigen-vectors, left and right, linearly independent. If D stands for the matrix whose columns are the right eigen-vectors of $A_n$, the necessary and sufficient condition for (1) to have a supplementary conservation law of the type

$$\partial_t h + \partial_i h^i = 0 \tag{2}$$

(*) Work supported by the C.N.R. through the «Gruppo Nazionale per la Fisica-Matematica».

(**) Istituto Matematico dell'Università di Messina.

(***) Nella seduta del 14 novembre 1974.

(1) The operators $\partial_t$ e $\partial_i$ are, respectively, the derivatives with respect to the time $t$ and to the space variables $x_i$ $(i = 1, 2, \cdots, m)$ · $\boldsymbol{f}^i$ and $\boldsymbol{u}$ are $m + 1$-column vectors of N components.

is that [1]:

$$(3)\qquad (\mathrm{D}\check{\mathrm{D}})^{-1} = \mathrm{H}\,(\boldsymbol{u})$$

be independent of $n_i$. In the expression (3), ˇ denotes the transposition operator and H is a symmetric positive definite square matrix. The $m+1$ scalar quantities $h$ and $h^i$ in (2) are given by

$$(4)\qquad \mathrm{H} = \nabla\nabla h$$

$$(5)\qquad h^i = \nabla\, h \boldsymbol{f}^i - g^i$$

$$(5')\qquad \nabla g^i = \check{\boldsymbol{f}}^i\,\mathrm{H}\,.$$

2. Consider a second order conservative hyperbolic equation in canonical form (2)

$$(6)\qquad \mathrm{U}_{tt} + \partial_i\,\mathrm{F}^i\,(\mathrm{U}_t\,,\mathrm{U}_k) = 0 \qquad (i = 1\,,2\,,3)\,;$$

by means of the substitution

$$(7)\qquad \mathrm{U}_t = \partial_t\,\mathrm{U} = \Phi \qquad \mathrm{U}_i = \partial_i\,\mathrm{U} = v^i$$

we obtain a first order system which can be written in the form (1') where

$$(8)\qquad \boldsymbol{u} = \begin{pmatrix} \Phi \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}. \qquad \mathrm{A}^i = \begin{pmatrix} a^i & b^i_k \\ -\delta^i_k & \underline{\underline{0}} \end{pmatrix}$$

$$(9)\qquad a^i = \frac{\partial \mathrm{F}^i}{\partial \mathrm{U}_t} \qquad b^i_k = \frac{\partial \mathrm{F}^i}{\partial \mathrm{U}_k}\,.$$

Of course, the solution of this system and of equation (6) are equivalent when we choose suitable initial conditions. The eigen-values of the matrix $\mathrm{A}_n$ defined by (3)

$$(10)\qquad \mathrm{A}_n = \mathrm{A}^i\,n_i = \begin{pmatrix} a_n & \boldsymbol{b}_n \\ -\boldsymbol{n} & \underline{\underline{0}} \end{pmatrix} \qquad (\boldsymbol{b}_n \equiv (b^k_i\,n_k))$$

are:

$$(11)\qquad \lambda_{1,2} = 0 \qquad \lambda_{3,4} = \frac{a_n \pm \sqrt{a_n - 4\,b_{nn}}}{2} \qquad (b_{nn} = \boldsymbol{b}_n \cdot \boldsymbol{n})$$

Corresponding to the double root $\lambda = 0$ we have the right eigen-vectors

$$(12)\qquad \boldsymbol{d}^{(1)} = \begin{pmatrix} 0 \\ \boldsymbol{w}_1 \end{pmatrix} \qquad \boldsymbol{d}^{(2)} = \begin{pmatrix} 0 \\ \boldsymbol{w}_2 \end{pmatrix}$$

with (4)

$$\boldsymbol{w}_1 \cdot \boldsymbol{b}_n = 0 \qquad \boldsymbol{w}_2 \cdot \boldsymbol{b}_n = 0 \qquad \boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \neq 0\,,$$

(2) $\mathrm{U}_{tt}$, of course, denotes the second derivative with respect to the time $t$.

(3) $\mathrm{A}_n$ is a matrix $4\times 4$; $\underline{\underline{0}}$ is a zero $3\times 3$ matrix.

(4) $\cdot$ and $\wedge$ are, respectively, scalar and vector products.

while for $\lambda = \lambda_3$, and $\lambda = \lambda_4$, we find, respectively,

$$d = \alpha \begin{pmatrix} \mu_1 b_{nn} \\ n \end{pmatrix} \qquad d = \beta \begin{pmatrix} \mu_2 b_{nn} \\ n \end{pmatrix} \tag{13}$$

with $\alpha$, $\beta$ arbitrary parameters and

$$\mu_1 = (\lambda_3 - a_n)^{-1} \qquad \mu_2 = (\lambda_4 - a_n)^{-1}. \tag{14}$$

In view of the relation

$$D\check{D} = \sum_{i=1}^{4} d^{(i)} \check{d}^{(i)} \tag{15}$$

from (12) and (13) we find

$$D\check{D} = \begin{pmatrix} (\alpha^2 \mu_1^2 + \beta^2 \mu_2^2) b_{nn}^2 & (\alpha^2 \mu_1 + \beta^2 \mu_2) b_{nn} n \\ (\alpha^2 \mu_1 + \beta^2 \mu_2) b_{nn} n & w_1 \otimes w_1 + w_2 \otimes w_2 + (\alpha^2 + \beta^2) n \otimes n \end{pmatrix}. \tag{16}$$

The hyperbolicity of the system requires the linear independence of the eigenvectors and hence, from (13), $b_{nn} \neq 0$. If we require $D\check{D}$ to be independent of $n$ then the following conditions must be satisfied

$$\alpha^2 \mu_1 + \beta^2 \mu_2 = 0 \tag{17}$$

$$(\alpha^2 \mu_1^2 + \beta^2 \mu_2^2) b_{nn}^2 = p^2(u) \tag{18}$$

$$w_1 \otimes w_1 + w_2 \otimes w_2 + (\alpha^2 + \beta^2) n \otimes n = T(u). \tag{19}$$

From (17) and (18) we obtain

$$\alpha^2 + \beta^2 = -\frac{p^2}{\mu_1 \mu_2 b_{nn}^2} = -\frac{p^2}{b_{nn}} \tag{20}$$

because (14) gives $\mu_1 \mu_2 = 1/b_{nn}$.

Moreover (16), together with (17), (18) and (19) becomes

$$D\check{D} = \begin{pmatrix} p^2(u) & 0 \\ 0 & T(u) \end{pmatrix}. \tag{21}$$

Equation (19) gives

$$T b_n = -p^2 n \tag{22}$$

which is valid for any $n$, so it follows that

$$TB = -p^2 I \qquad (B = \| b_k^i \|) \tag{23}$$

and we then have

$$(D\check{D})^{-1} = 1/p^2 \begin{pmatrix} I & 0 \\ 0 & -B \end{pmatrix}. \tag{24}$$

3. In this case we have

$$H = \nabla \check{\nabla} h = \begin{pmatrix} \dfrac{\partial^2 h}{\partial \Phi^2} & \dfrac{\partial^2 h}{\partial \Phi \, \partial v^i} \\ \dfrac{\partial^2 h}{\partial v^i \, \partial \Phi} & \dfrac{\partial^2 h}{\partial v^i \, \partial v^k} \end{pmatrix} \tag{25}$$

while from (3) and (24) we obtain the following differential system for $h$:

$$\begin{aligned}(26)\qquad & \partial^2 h/\partial\Phi^2 = 1/p^2 \\ & \partial^2 h/\partial\Phi\,\partial v_i = 0 \\ & \partial^2 h/\partial v_i\,\partial v_k = (-1/p^2)\, b_{ik} = (-1/p^2)\,\partial F^i/\partial v_k\,.\end{aligned}$$

From $(26)_2$ we have, immediately, that

$$(27)\qquad h = h_1(\Phi) + h_2(\boldsymbol{v})$$

while $(26)_1$, $(26)_2$ then give, respectively

$$(28)\qquad 1/p^2 = h_1''$$

$$(29)\qquad F^i = (-1/h_1'')\,[\partial h_2/\partial v_i + \psi^i(\Phi)]\,.$$

As $\boldsymbol{f}^i$ is given by

$$(30)\qquad \boldsymbol{f}^i = \begin{pmatrix} F^i \\ -\delta^i_k\,\Phi \end{pmatrix}$$

(5′) requires that $g^i$ satisfies the equations

$$(31)\qquad \partial g^i/\partial\Phi = (1/p^2)\,F^i$$

$$(32)\qquad \partial g^i/\partial v_k = (\Phi/p^2)\,b^i_k = (\Phi/p^2)\,\partial F^i/\partial v_k$$

and, because of (28) and (29), we find that

$$(33)\qquad g^i = -\Phi\,(\mathrm{d}h_2/\mathrm{d}v^i) + \int \psi^i(\Phi)\,\mathrm{d}\Phi\,.$$

Moreover, in view of equations (28), (29) we have

$$(34)\qquad \nabla h\boldsymbol{f}^i = h_1'\,F^i - \int \psi^i(\Phi)\,\mathrm{d}\Phi\,.$$

and so, on account of (33), from (5) we obtain

$$(35)\qquad h^i = h_1'\,F^i - \int \psi^i(\Phi)\,\mathrm{d}\Phi\,.$$

Now the equations (27) and (35) allow us to write the conservation law (2) in the form

$$(36)\qquad \partial_t(h_1 + h_2) + \partial_i\left\{-\int \psi^i\,\mathrm{d}\Phi - (h_1'/h_1'')\,(\mathrm{d}h_2/\mathrm{d}v^i + \psi^i)\right\} = 0\,.$$

Finally if we use (29) in (6) we arrive to the result

$$\begin{aligned}(37)\qquad & U_{tt} + (1/h_1'')^2\,\{h_1'''\,(\mathrm{d}h_2/\mathrm{d}v^i) - h_1'''\,\psi^i + h_1''\,\psi'^i\}\,U_{ti} + \\ & -(1/h_1'')(\mathrm{d}^2 h_2/\mathrm{d}v^i\,\mathrm{d}v^k)\,U_{ik} = 0\,.\end{aligned}$$

This is the most general second order equation admitting a conservation law of the type (36). It is easy to prove that the results are available also in a $m$-dimensional space ($i = 1, 2, \cdots, m$).

Let us consider the particular case of a non-linear string equation [2]:

$$U_{tt} = c^2(1 + kU_x^p)\,U_{xx} \qquad (k \text{ and } p \text{ are constants}) \tag{38}$$

that in conservative form becomes

$$U_{tt} + \partial_x\,[(-c^2)\,(U_x + k/(p+1)\,U_x^{p+1})] = 0. \tag{39}$$

If we compare equation (38) with (37), we have

$$\begin{cases} h_1'' = 1 \\ \dfrac{d^2 h_2}{dU_x^2} = c^2\,(1 + kU_x^p) \\ \psi'^1 = 0 \Rightarrow \psi^1 = \text{const.} \end{cases} \tag{40}$$

Consequently by integration of (40) we obtain

$$\begin{cases} h_1 = \frac{1}{2}\,\Phi^2 \\ \dfrac{dh_2}{dU_x} = -F = c^2\,[U_x + k/(p+1)\,U_x^{p+1}] \\ h_2 = c^2\,[\frac{1}{2}\,U_x^2 + k/(p+1)\,(p+2)\,U_x^{p+2}] + \psi_0\,(\Phi) \end{cases} \tag{41}$$

where we have set equal to zero the constants of integration. So, from (35), it follows:

$$h^i = -\Phi F \delta_1^i\,. \tag{42}$$

Then, because of (41) and (42), we can write the conservation law given by (36).

4. In this last part we look directly for a conservation law of type (2) for system (8). We suppose that there is the following supplementary conservative law for the system (8):

$$\partial_t\,h + \partial_i\,h^i = 0 \tag{43}$$

or

$$\frac{\partial h}{\partial \Phi}\,\Phi_t + \frac{\partial h}{\partial v^i}\,\Phi_i + \frac{\partial h^i}{\partial \Phi}\,\Phi_i + \frac{\partial h^i}{\partial v^k}\,v^k_{,i} = 0 \tag{44}$$

and we seek the condition which must be satisfied by $h$ and $h^i$. Taking into account system (8), eq. (44) becomes

$$\left\{\frac{\partial h}{\partial v^i} + \frac{\partial h^i}{\partial \Phi} - \frac{\partial h}{\partial \Phi}\,\frac{\partial F^i}{\partial \Phi}\right\}\Phi_i + \left\{\frac{\partial h^i}{\partial v^k} - \frac{\partial h}{\partial \Phi}\,\frac{\partial F^i}{\partial v^k}\right\}v^k_{,i} = 0. \tag{45}$$

If we now impose the condition that this equation has to be identically satisfied with respect to $v_{i,k}$ and $\Phi_i$ we arrive to the differential system

$$\frac{\partial h^i}{\partial v^k} - \frac{\partial h}{\partial \Phi}\,\frac{\partial F^i}{\partial v^k} = 0 \tag{46}$$

$$\frac{\partial h}{\partial v^i} + \frac{\partial h^i}{\partial \Phi} - \frac{\partial h}{\partial \Phi}\,\frac{\partial F^i}{\partial \Phi} = 0\,. \tag{47}$$

By differentiating eq. (46) with respect to $\Phi$ and eq. (47) with respect to $v^k$ we obtain the compatibility condition

$$\frac{\partial^2 h}{\partial v^k \partial \Phi} \frac{\partial F^i}{\partial \Phi} - \frac{\partial^2 h}{\partial v^i \partial v^k} - \frac{\partial^2 h}{\partial \Phi^2} \frac{\partial F^i}{\partial v^k} = 0 . \tag{48}$$

Moreover eq. (46) differentiated with respect to $v^j$ gives

$$\frac{\partial^2 h}{\partial v_k \partial v^j} = \frac{\partial^2 h}{\partial \Phi \partial v^j} \frac{\partial F^i}{\partial v^k} + \frac{\partial h}{\partial \Phi} \frac{\partial^2 f^i}{\partial v^j \partial v^k} ; \tag{49}$$

and, interchanging $j$ and $k$ then gives the following relation

$$\frac{\partial^2 h}{\partial \Phi \partial v^j} \frac{\partial F^i}{\partial v^k} = \frac{\partial^2 h}{\partial \Phi \partial v^k} \frac{\partial F^i}{\partial v^j} . \tag{50}$$

If we suppose that at least one of the component of the vector $\partial^2 h/\partial \Phi \partial v^k$, for instance when $k = 1$, is different from zero, then we can write

$$\frac{\partial^2 h}{\partial \Phi \partial v^j} \varphi^i = \frac{\partial F^i}{\partial v^j} \qquad \text{with} \quad \varphi^i = \frac{\partial F^i}{\partial v_1} \Big/ \frac{\partial^2 h}{\partial \Phi \partial v_1} \tag{51}$$

so (48) becomes

$$\frac{\partial^2 h}{\partial v^i \partial v^k} = \gamma^i \Gamma^k \tag{52}$$

where

$$\gamma^i = \frac{\partial F^i}{\partial \Phi} - \frac{\partial^2 h}{\partial \Phi^2} \varphi^i \tag{53}$$

$$\Gamma^k = \frac{\partial^2 h}{\partial v^k \partial \Phi} . \tag{54}$$

We impose now the condition that the matrix H, given by (4), is positive definite, that is

$$\check{\boldsymbol{v}} \mathrm{H} \boldsymbol{v} > 0 \qquad \text{with} \quad \boldsymbol{v} = \begin{pmatrix} \omega \\ \eta_i \end{pmatrix} \tag{55}$$

and so

$$\frac{\partial^2 h}{\partial \Phi^2} = p^2 > 0 \tag{56}$$

$$\Delta = \left( \frac{\partial^2 h}{\partial \Phi \partial v^i} \eta_i \right)^2 - p^2 \frac{\partial^2 h}{\partial v^i \partial v^k} \eta_i \eta_k < 0 . \tag{57}$$

Eq. (57) together with (52) and (54), gives

$$\Delta = \Gamma_k \eta_k \{ \Gamma_i \eta_i - p^2 \gamma_i \eta_i \} < 0 \tag{58}$$

which must be satisfied for any $\eta_i$. If we choose, for instance, $\eta \perp \boldsymbol{\Gamma}$ we have $\Delta = 0$ and the root $\omega = 0$ is in contradiction with the hypothesis that H is strictly positive. Consequently, it is necessary to impose the condition

$$\frac{\partial^2 h}{\partial v^k \partial \Phi} = 0 \qquad \forall k \tag{59}$$

and we have again

$$h = h_1(\Phi) + h_2(\boldsymbol{v}). \tag{60}$$

From eq. (48), together with (59) and (60) we deduce

$$\frac{\partial^2 h_2}{\partial v^i \, \partial v^k} + \frac{\partial^2 h_1}{\partial \Phi^2} \frac{\partial F^i}{\partial v^k} = 0 \tag{61}$$

and by integration

$$\frac{\partial h_2}{\partial v^i} + h_1'' \, F^i = \psi^i(\Phi). \tag{62}$$

From (46), again by integration, we have

$$h^i = h_1' \, F^i + \theta^i(\Phi) \tag{63}$$

where $\theta^i$ by virtue of eqts. (47) and (62) is given by

$$\theta^i(\Phi) = -\int \psi^i(\Phi) \, d\Phi. \tag{64}$$

We can conclude that by adopting the hypothesis that H is strictly positive we obtain the same results as in section 3.

## References

[1] G. Boillat (1974) – *Sur l'existence et la recherche d'equations de conservation supplémentaires pour les systémes hyperboliques*, «Comptes Rendus», *278*, sér. A, 909.

[2] A. Jeffrey and T. Kakutani (1972) – *Weak non-linear dispersive waves: a discussion centered around the Kortweg-De Vries equation*, «Siam Rievew», *14* (4), October, 583.