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Topologies generated by orders and quasi-uniform structures

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Topologia. — *Topologies generated by orders and quasi-uniform structures.* Nota di JOHN W. CARLSON, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si stabilisce una stretta relazione fra topologie generate da un ordine e strutture quasi uniformi, mostrando come le seconde forniscano uno strumento assai utile nello studio delle prime.

1. INTRODUCTION

The purpose of this paper is twofold. The primary purpose is to characterize those topologies which are generated by a single partial order or quasi-order and then to characterize those topologies that are generated by a family of quasi-orders. The second purpose is more expository in nature, namely to convey the close relationship between the notions of a topology generated by an order or a family of orders in some sense and the concepts of quasi-uniform structures. That this is completely natural is evident from the definitions. It is noted that quasi-uniform structures provide a useful tool in the study of topologies generated by an order relationship.

DEFINITION 1.1. A quasi-order on X is a reflexive and transitive relation. A partial order is an antisymmetric quasi-order. A linear order is a totally ordered partial order. A family $\mathcal{P}$ of quasi-orderings on a set X is a filter base of quasi-orderings on X. Thus,

(1) $\leq_\alpha \in \mathcal{P}$, $\leq_\beta \in \mathcal{P}$ implies $\leq_\gamma \in \mathcal{P}$ where $x \leq_\gamma y$ if and only if $x \leq_\alpha y$ and $x \leq_\beta y$.

DEFINITION 1.2. A topology t on a set X is generated by a family $\mathcal{P}$ of quasi-orderings provided:

$$t = \{o : \text{if } x \in o \text{ then there exists } \leq_\alpha \in \mathcal{P} \text{ such that } x \leq_\alpha y \text{ implies } y \in o\}.$$

This is called the segment topology generated by the family $\mathcal{P}$.

DEFINITION. 1.3. Let X be a non-empty set. A quasi-uniform structure for X is a filter $\mathcal{U}$ of subsets of $X \times X$ such that;

(1) $\Delta = \{(x, x) : x \in X\} \subset U$ for each U in $\mathcal{U}$, and

(2) for each U in $\mathcal{U}$ there exists a V in $\mathcal{U}$ with $V \circ V \subset U$.

DEFINITION 1.4. If $\mathcal{U}$ is a quasi-uniform structure for a set X, let

$$t_{\mathcal{U}} = \{o \subset X : \text{ if } x \in o \text{ there exists U in } \mathcal{U} \text{ such that } U[x] \subset o\}.$$

Then $t_{\mathcal{U}}$ is the quasi-uniform topology on X generated by $\mathcal{U}$. A quasi-uniform structure $\mathcal{U}$ is said to be compatible with a topology t provided $t = t_{\mathcal{U}}$.

(*) Nella seduta del 14 novembre 1974.

In [4], Pervin provided a useful construction of a compatible quasi-uniform structure for a given topology. We call his structure the Pervin structure for (X, t) and it is given by the subbase

$$\mathscr{S} = \{o \times o \cup (X - o) \times X : o \in t\}.$$

A base $\mathscr{B}$ for a quasi-uniform structure is called transitive if $B \circ B = B$ for each B in $\mathscr{B}$. A quasi-uniform structure is called a transitive quasi-uniform structure if it has a transitive base. It is well-known that the Pervin structure is transitive. An excellent introduction to quasi-uniform spaces may be found in [5].

A topology is called saturated if the arbitrary intersection of open sets is open. A quasi-uniform structure $\mathscr{U}$ is called saturated provided $\mathscr{U}$ is closed under arbitrary intersections. Note that $\mathscr{U}$ is saturated if and only if it has a base consisting of a single set.

2. Partial orderings

The first theorem characterizes those topologies that are generated by a single partial order with the segment topology.

THEOREM 2.1. *Let (X, t) be a topological space. t is generated by a partial order, with the segment topology, if and only if there exists a compatible saturated quasi-uniform structure $\mathscr{U}$ generated by a base $\{U\}$ with $U \cap U^{-1} = \Delta$.*

Proof. Let t be generated by the partial order $\leq$. Set $U = \{(x, y) : x \leq y\}$. Then $U \circ U = U$ and since $\leq$ is antisymmetric $U \cap U^{-1} = \Delta$. Now $\{U\}$ is a base for a saturated quasi-uniform structure $\mathscr{U}$ and $t = t_{\mathscr{U}}$. Conversely, define $x \leq y$ if and only if $(x, y) \in U$. Then t is generated by this partial order.

We now consider totally ordered partial orders or linear orders. Thron and Zimmerman have characterized those topologies that are generated by a linear order with the interval topology. Their result, found in [7], is the following theorem.

THEOREM A. *A topology t on a set X is an order topology if and only if (X, t) is a T_1 space and t is the least upper bound of two minimal T_0 topologies.*

Since we are attempting to stress the close relationship between the study of topologies generated by orders and the study of quasi-uniform structures we present the development of the above theorem in terms of quasi-uniform structures. The proofs are natural and omitted.

LEMMA 2.1. *Let $\{U\}$ generate a quasi-uniform structure $\mathscr{U}$. Then o belongs to $t_{\mathscr{U}}$ if and only if for each x in o there exists y in X with $x \in U[y] \subset o$.*

LEMMA 2.2. *Let $\mathscr{U}$ be a saturated quasi-uniform structure generated by the base $\{U\}$ on X such that $U \cup U^{-1} = X \times X$ and $U \cap U^{-1} = \Delta$. Set $U' = U - \Delta$. Let $o \in t'_{\mathscr{U}}$ if and only if $o = X$ or for each x in o there exists y in X with $x \in U'[y] \subset o$. Then $t'_{\mathscr{U}}$ is a topology on X.*

LEMMA 2.3. *A topology t is minimal* T_0 *if and only if there exists a saturated quasi-uniform structure* $\mathcal{U}$ *generated by a base* $\{U\}$ *with* $U \cup U^{-1} = X \times X$ *and* $U \cap U^{-1} = \Delta$, *such that* $t'_{\mathcal{U}} = t$.

THEOREM 2.2. *A topology t on a set* X *is an order topology if and only if* X *is* T_1, *and there exists a saturated quasi-uniform structure* $\mathcal{U}$ *generated by a base* $\{U\}$ *with* $U \cup U^{-1} = X \times X$ *and* $U \cap U^{-1} = \Delta$ *such that* $t = t'_{\mathcal{U}} \vee t'_{\mathcal{U}^{-1}}$.

The question as to which topological spaces are generated by a family of partial orders with the segment topology is unanswered. One notes that if a topology is generated in this manner then it must be T_0. It can be shown that the usual topology on the reals is not generated in this way. Every uniform structure that has a transitive antisymmetric base must produce the discrete topology. This question stated in terms of quasi-uniform structures is the following.

QUESTION. Which topological spaces admit a compatible quasi-uniform structure that has a transitive antisymmetric base?

3. QUASI-ORDERINGS

Our first theorem shows that precisely the saturated topologies are the ones that are generated by a single quasi-ordering.

THEOREM 3.1. *Let* (X, t) *be a topological space. t is generated by a single quasi-ordering if and only if t admits a saturated quasi-uniform structure.*

The proof of this theorem is similar to the proof of Theorem 2.1.

The existence of a one-to-one correspondence between quasi-orders and topologies on a finite set has been noted by Fletcher in [3] and by Sharp in [6]. Since every finite topological space is saturated we have the following result.

COROLLARY 3.2. *Let* (X, t) *be a finite topological space. Then there exists a quasi-ordering on* X *which generates the topology.*

It is clear that every finite family of quasi-orderings $\mathcal{P}$ on a set X can be replaced with a singleton family $\mathcal{P}'$ such that $\mathcal{P}$ and $\mathcal{P}'$ generate the same topology.

Example 3.1. The following shows that the procedure used to characterize the interval topology of a linear order does not lend itself to characterizing the interval topology of a totally ordered quasi-order. The closed interval topology of a totally ordered partial order generates the discrete topology. Example 3.1 demonstrates that this need not be the case for a totally ordered quasi-order.

Let $X = \{1, 2, 3, 4\}$ and $R = \Delta \cup U'$ where

$$U' = \{(1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 4)\}.$$

Then R is a totally ordered quasi-order. R is not antisymmetric, so condition 2 of Lemma 2.2 does not hold. Also, $U'[1] = \{3, 4\}$, $U'[2] = \{1, 3, 4\}$, $U'[3] = \{1, 4\}$, $U'[4] = \varnothing$. Now $t'_{\mathscr{U}} = \{\varnothing, X, \{3, 4\}, \{1, 3, 4\}, \{1, 4\}\}$ but $t'_{\mathscr{U}}$ is not a topology since $\{3, 4\} \cap \{1, 4\} \notin t'_{\mathscr{U}}$. Thus Lemma 2.2 need not hold if the antisymmetric condition is deleted. Now the base for the closed interval topology generated by R is $\{\{1, 3\}, \{2\}, \{4\}\}$, and thus the topology is not discrete. Set $U = R \cap R^{-1}$. Then $\{U\}$ forms a base for a quasi-uniform structure that generates the above topology.

THEOREM 3.3. *Let* X *be a finite set and* R *a totally ordered quasi-order on* X. *Let t denote the closed interval topology on* X. (*A subbase for t is* X *and all sets of the form* $\{y : y \leq c\}$ *or* $\{y : y \geq c\}$ *for* $c \in X$, *where* $a \leq b$ *if and only if* $(a, b) \in R$). *For each* x *in* X, *set* $O_x = \cap \{o : x \in o \in t\}$ *and* $U = \{(x, y) : y \in O_x\}$. *Then* (1) $U = R \cap R^{-1}$, (2) $\{U\}$ *forms a base for a quasi-uniform structure* $\mathscr{U}$ (3) $t = t_{\mathscr{U}}$, *and* (4) t *is completely regular.*

Proof. (1) We observe that for each x in X, $O_x = \{t : x \leq t \leq x\}$. Suppose $(a, b) \in U$, then $b \in O_a$ and $a \leq b \leq a$. Thus $(a, b) \in R$ and $(a, b) \in R^{-1}$. Suppose $(a, b) \in R \cap R^{-1}$, then $a \leq b \leq a$ and $b \in O_a = U[a]$. Hence $(a, b) \in U$ and $U = R \cap R^{-1}$.

(2) Since R is reflexive, we have that $\Delta \subset R \cap R^{-1} = U$. Also R and R^{-1} are transitive, thus $U \circ U = U$. Thus $\{U\}$ forms a base for a quasi-uniform structure, which we denote by $\mathscr{U}$.

(3) Let $o \in t$ and $x \in o$. Then $x \in O_x \subset o$, but $O_x = U[x]$. Hence $t \leq t_{\mathscr{U}}$. Let $x \in o \in t_{\mathscr{U}}$, then $x \in U[x] \subset o$. That is $x \in O_x \subset o$ but $O_x \in t$. Therefore $o \in t$ and $t = t_{\mathscr{U}}$.

(4) Since U is symmetric $\mathscr{U}$ is a uniform structure and thus $t_{\mathscr{U}} = t$ is uniformizable or completely regular.

It is natural to ask which topologies are generated by a family of quasi-orderings. Our next theorem provides the answer.

THEOREM 3.4. *Let* (X, t) *be a topological space.* t *is generated by a family of quasi-orderings if and only if* (X, t) *admits a compatible transitive quasi-uniform structure.*

Proof. Let $\mathscr{P} = \{\leq_\alpha : \alpha \in \mathscr{L}\}$ be a family of quasi-orderings which generates t. Set $\mathscr{B} = \{U_\alpha : \alpha \in \mathscr{L}\}$, where $U_\alpha = \{(x, y) : x \leq_\alpha y\}$. Since each quasi-ordering $\leq_\alpha$ is transitive we have that $U_\alpha \circ U_\alpha = U_\alpha$. Let $U_\alpha, U_\beta \in \mathscr{B}$, then there exists $U_\gamma \in \mathscr{B}$ and $U_\gamma \subset U_\alpha \cap U_\beta$, where $x \leq_\gamma y$ if and only if $x \leq_\alpha y$ and $x \leq_\beta y$. Now $\mathscr{B}$ forms a base for a quasi-uniform structure $\mathscr{U}$ and $t = t_{\mathscr{U}}$.

Conversely, if (X, t) admits a transitive structure $\mathscr{U}$, then $\mathscr{U}$ has a base $\mathscr{B} = \{U_\alpha : \alpha \in \mathscr{L}\}$ and $U_\alpha \circ U_\alpha = U_\alpha$. Define $\mathscr{P}$ by $\mathscr{P} = \{\leq_\alpha : \alpha \in \mathscr{L}\}$ where $x \leq_\alpha y$ if and only if $(x, y) \in U_\alpha$. Then $\mathscr{P}$ is a family of quasi-orderings on X and $\mathscr{P}$ generates the topology t.

Since the Pervin structure has a transitive base we have the following corollary.

COROLLARY 3.5. *Every topology is generated by a family of quasi-orderings.*

It is unnecessary to require each order in the family to be transitive provided the family is transitive in the sense that given $\leq_\alpha \in \mathcal{P}$ then there exists $\leq_\beta$ in $\mathcal{P}$ such that $x \leq_\beta y$ and $y \leq_\beta z$ implies that $x \leq_\alpha z$. Every topology is generated by a transitive family of reflexive orders since every topology admits a compatible quasi-uniform structure. In this sense the segment topology is not as restrictive as one might suppose. Consider the reflexive order $U(\varepsilon) = \{(x, y) : x - \varepsilon < y < x + \varepsilon\}$ on the reals. Then the segment $\{y : a \leq_{U(\varepsilon)} y\}$ is just the usual interval $(a - \varepsilon, a + \varepsilon)$.

From the proof of Theorem 3.4 it is evident that we have the following.

THEOREM 3.6. *Let* (X, t) *be a topological space.* t *is generated by a countable family of quasi-orderings if and only if* (X, t) *admits a transitive quasi-uniform structure with a countable base.*

This theorem together with Theorems 3.2 and 3.4 of [2] yields the following corollaries.

COROLLARY 3.7. (X, t) *is generated by a countable family of quasi-orderings if and only if* (X, t) *is generated by a non-Archimedian quasi-metric.*

COROLLARY 3.8. *Every orthocompact Moore space is generated by a countable family of quasi-orderings.*

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