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Alexandru Neagu

On the intersection of principal fibre subbundle

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Geometria differenziale. — *On the intersection of principal fibre subbundle.* Nota di ALEXANDRU NEAGU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Questo lavoro verte su qualche problema riguardante l'intersezione dei sottofibrati principali chiusi di uno spazio fibrato principale differenziabile.

Let $P(M, G)$ be a principal differentiable fibre bundle. We denote by π the canonical projection $P \to M$ and let $\mathscr{A} = \{(U_i, \varphi_i)/i \in I\}$ be the highest atlas of $P(M, G)$, where (U_i, φ_i) are allowable charts of P.

A principal fibre bundle $P_1(M, G_1)$ is a principal subbundle of $P(M, G)$ if:

a) P_1 is a submanifold of P, and G_1 is a Lie subgroup of G;

b) $\pi_1 = \pi|_{P_1}$, where π_1 is the projection $P_1 \to M$;

c) $\bar{R}_g = R_g|_{P_1}$, where $\bar{R}_g$ and R_g are translations on P_1 and P respectively, defined by $g \in G_1$.

PROPOSITION 1 [1]. *The subset* $P_1 \subset P(M, G)$ *is a principal subbundle of* $P(M, G)$ *if, and only if,* $\pi_1 = \pi|_{P_1}$ *satisfies the following conditions*:

a) $\pi_1(P_1) = M$;

b) $\pi_1^{-1}(x) = z \cdot G_1$ if $z \in \pi_1^{-1}(x)$ *and* $x = \pi(z)$;

c) for every point $x \in M$ *there exist an open neighborhood* U *of* x *and a differentiable mapping* $\sigma : U \to P(M, G)$ *satisfying* $\sigma(U) \subset P_1$ *and* $\pi_1 \circ \sigma = id_U$.

PROPOSITION 2 [3]. $P_1(M, G_1)$ *is a closed subbundle of* $P(M, G)$ *if and only if there exists a cross section* $s : M \to P/G_1$.

LEMMA 1. *The structure group* G *of* $P(M, G)$ *is reducible to a closed subgroup* $G_1 \subset G$ *if, and only if, the following conditions are satisfied*:

a) *there exist a differentiable manifold* V, *a representation of* G *on* V, $(g, u) \in G \times V \to g \cdot u \in V$, *and a point* $u_0 \in V$ *such that the isotropy group of* u_0 *is* G_1. *The orbital mapping* $\rho(u_0) : G \to V$ *defined by* $\rho(u_0) \cdot a = a \cdot u_0$ *is a subimmersion* (this condition is obviously satisfied in the finite dimensional case);

b) there exists a morphism $A : P \to V$ *such that* $A(P) = Gu_0$ *(the orbit of* u_0*) and* $A(z \cdot g) = g^{-1} A(z)$ *for every* $z \in P$ *and* $g \in G$.

Proof. Let us consider the map

$$i_{u_0} : \; a/G \in G/G_1 \to i_{u_0}(a/G_1) = au_0 \in V .$$

(*) Nella seduta del 14 novembre 1974.

We prove that i_{u_0} is an immersion. If $a/G_1 \neq b/G_1$ then $a^{-1} \cdot b \notin G_1$. Supposing $i_{u_0}(a/G_1) = i_{u_0}(b/G_1)$ it results $au_0 = bu_0$ and $u_0 = a^{-1} bu_0$, in other words $a^{-1} b \in G_1$. Let λ be the canonical projection $G \to G/G_1$. We have $\rho(u_0) = i_{u_0} \circ \lambda$ and $\rho(u_0) \cdot a = (i_{u_0} \circ \lambda)(a) = i_{u_0}(a/G_1) = a \cdot u_0$. Since $\rho(u_0)$ and λ are analytic it follows that i_{u_0} is analytic. Since gG_1 is a submanifold of G then $T_g(gG_1) = \text{Ker. } T_g(\rho(u_0))$; but $\text{Ker } T_g \lambda = T_g(gG_1)$ and hence $T_{\lambda(g)} i_{u_0}$ is injective. On the other hand, the image of $T_{\lambda(g)} i_{u_0}$ coincides with the image of $T_g(\rho(u_0))$, and the latter has a topological supplement. Consequently i_{u_0} is an immersion.

We shall prove now the first statement of the lemma. Let π_1 be the restriction of π to $A^{-1}(u_0)$. We prove that π_1 satisfies the conditions of Proposition 1. Since Gu_0 is the immersed submanifold of V, it results that A is a morphism of P on $Gu_0 = G/G_1$. Assume $z_1 \in \pi^{-1}(x) \subset P$. Since $A(z_1) \in Gu_0$ there is $g \in G$ such that $A(z_1) = gu_0$ and:

$$A(z_1 \cdot g) = g^{-1} A(z_1) = g^{-1} gu_0 = u_0 .$$

It follows that $z_1 g \in A^{-1}(u_0)$, so that $\pi_1(A^{-1}(u_0)) = M$.

Let z_1 and z_2 be two points of $A^{-1}(u_0)$ satisfying $\pi_1(z_1) = \pi_1(z_2) = x$ and $g \in G$ such that $z_2 = z_1 \cdot g$. It follows that $A(z_2) = A(z_1 \cdot g) = g^{-1} A(z_1)$, hence $g^{-1} u_0 = u_0$ and $g \in G_1$. Accordingly $\pi_1^{-1}(x) = z_1 G$.

Let U' be an open neighbourhood in G/G_1 and $W = i_{u_0}(U')$. Then W is an open set in Gu_0 equipped with the induced topology, and $A^{-1}(W)$ is open in P. Let $U \subset A^{-1}(W)$ be an open set of P. We have $A(U) \subset W$ and $i_{u_0}^{-1}(A(U)) \subset U'$. It is clear that for every open set U' in G/G_1, there is an open set U in P such that $(i_{u_0}^{-1} \circ A)(U) \subset U'$. Let τ be a local cross-section over U'; we have:

$$U \subset M \xrightarrow{s} P(M, G) \xrightarrow{A} Gu_0 \xrightarrow{i_{u_0}^{-1}} G/G_1 \xrightarrow{\tau} G.$$

If λ is the canonical projection $G \to G/G_1$ then $\lambda \circ \tau = id$. Let us denote $\sigma = i_{u_0}^{-1} \circ A \circ s$, $h = \tau \circ \sigma$ and $\eta(x) = s(x) \cdot h(x)$ (for $x \in U$). Then $\lambda \circ h = \lambda \circ \tau \circ \sigma = \sigma$ and

$$\begin{aligned} A(\eta(x)) &= A(s(x) \cdot h(x)) = [h(x)]^{-1} \cdot A(s(x)) = [h(x)]^{-1} \cdot (i_{u_0} \circ \sigma)(x) = \\ &= [h(x)]^{-1} \cdot i_{u_0}((\lambda \circ h)(x)) = [h(x)]^{-1} \cdot i_{u_0}(h(x)/G_1) = \\ &= [h(x)]^{-1} \cdot h(x) \cdot u_0 = u_0 . \end{aligned}$$

Hence η is a local cross-section over U, with its values in $A^{-1}(u_0)$.

Conversely, let $P_1(M, G_1)$ be a closed principal fibred subbundle of $P(M, G)$ and $V = G/G_1$. Then there is a global cross-section $s : M \to P/G_1$. Let (U, φ) and (U, ψ) be the bundles charts of $P(M, G)$ and P/G_1, respectively.

One can define the morphism $A : P \to V$ by:

$$A(z) = [\varphi_x^{-1}(x)]^{-1} \cdot \psi_x^{-1}(s(x)) \qquad \text{for } z \in \pi^{-1}(x) \text{ and } x \in U .$$

Here φ_x (resp. ψ_x) is the restriction of φ (resp. ψ) to $\{x\}\times G$ (resp. $\{x\}\times G/G_1$). If $(\bar{U}, \bar{\varphi})$ and $(\bar{U}, \bar{\psi})$ are two associated bundles charts such that $U \cap \bar{U} \neq \varnothing$ then

$$[\bar{\varphi}_x^{-1}(z)]^{-1}\,\bar{\psi}_x^{-1}(s(x)) = [a_{\bar{\varphi}\varphi}(x)\,\varphi_x^{-1}(z)]^{-1}\,[a_{\bar{\psi}\psi}(x)\,\psi_x^{-1}](s(x)) =$$

$$= [\varphi_x^{-1}(z)]^{-1}\,[a_{\bar{\varphi}\varphi}(x)]^{-1}\,a_{\bar{\psi}\psi}(x)\,\psi_x^{-1}(s(x)) = [\varphi_x^{-1}(z)]^{-1}\,\psi_x^{-1}(s(x)).$$

Where $a_{\bar{\varphi}\varphi}$ (resp. $a_{\bar{\psi}\psi}$) is the transition function subordinate of charts (U, φ) and $(\bar{U}, \bar{\varphi})$ (resp. $(U, \bar{\psi})$ and $(\bar{U}, \psi)$). Here we have used the property: $a_{\bar{\varphi}\varphi}(x) = a_{\bar{\psi}\psi}(x)$ for the associated bundles charts. Assume $z \in P$ and $g \in G$. Thus

$$A(z \cdot g) = A[\varphi_x(\varphi_x^{-1}(z) \cdot g)] = [\varphi_x^{-1}(\varphi_x(\varphi_x^{-1}(z)) \cdot g]^{-1} \cdot \psi_x^{-1}(s(x)) =$$

$$= g^{-1} \cdot [\varphi_x^{-1}(z)]^{-1}\,\varphi_x^{-1}(s(x)) = g^{-1}\,A(z). \qquad q.e.d.$$

Consequence 1. In the conditions of the Lemma 1, if $u_1, u_2 \in Gu_0$ are the isotropy groups G_1 and G_2 respectively, then $A^{-1}(u_1)$ and $A^{-1}(u_2)$ are conjugated subbundles; more precisely there is $g \in G$ such that $A^{-1}(u_1) \cdot g = A^{-1}(u_2)$ and $G_2 = g^{-1}\,G_1$.

Indeed, let a be the element of G such that $u_2 = a \cdot u_1$; we have $A(z \cdot a^{-1}) = a \cdot A(z) = au_1 = u_2$ for every $z \in A^{-1}(u_1)$. Then $A^{-1}(u_1) \cdot a^{-1} = A^{-1}(u_2)$, and so the assertion is true for $g = a^{-1}$.

Consequence 2. In the conditions of the Consequence 1, $G = G_1$ if and only if $g \in \mathcal{N}(G_1)$ (the normalizer of G_1 in G) or $g \in \mathcal{N}(G_2)$ (the normalizer of G_2 in G).

THEOREM 1. *Let* $P(M, G)$ *be a principal fibred bundle and* $P_1(M, G_1)$, $P_2(M, G_2)$ *two closed subbundles of* $P(M, G)$ *such that* $G_1 \cap G_2$ *is a Lie subgroup of* G. *The intersection* $P_1 \cap P_2$ *is a subbundle of* P *if and only if*, $\pi(P_1 \cap P_2) = M$, *where* π *is the projection of* $P(M, G)$.

Proof. We have the morphism $A_1 : P \to G/G_1$, which satisfies the conditions *a)* and *b)* of Lemma 1, and $A_1^{-1}(e/G_1) = P_1(M, G_1)$. The group G_2 acts on G/G_1 and the isotropy group of $u_0 = e/G_1$ is $G_1 \cap G_2$. Let A be the restriction of A_1 to P_2. Let π_1 and π_2 be the restrictions of π to P_1 and P_2 respectively. Let α be the fixed point of $\pi_1^{-1}(x) \cap \pi_2^{-1}(x)$ for any $x \in M$; then for every $\beta \in P$ there is $g \in G_2$ such that $\beta = \alpha \cdot g$. We have:

$$A(\beta) = A(\alpha \cdot g) = g^{-1}\,A(\alpha) = g^{-1} \cdot e/G_1 = g^{-1}\,u_0 \in G_2\,u_0\,.$$

Then the morphism A takes its values in the orbit $G_2\,u_0$. It follows that $P_2(M, G_2)$ is reducible to a subgroup $G_2 \cap G_1$, and the reduced bundle is $A^{-1}(u_0) = P_1 \cap P_2$.

Example 1. Let Δ^1 and Δ^2 be two distributions on a manifold M, where $\dim M = n$, $\dim \Delta^1 = p_1$ and $\dim \Delta^2 = p_2$. Let G_α be the subgroups of

GL (n, R) defined by:

$$G_\alpha = \{ \| a^i_j \| \in GL(n, R) / a^{i'_\alpha}_{b_\alpha} = 0 \}$$

$$b_\alpha = 1, 2, \cdots, p_\alpha \quad ; \quad i'_\alpha = p_\alpha + 1, \cdots, n \quad ; \quad \alpha = 1, 2.$$

Let $\mathcal{F}(M)$ denote the principal fibred bundle of all linear frames of M and let $\mathcal{G}^{p_\alpha}(M) = \mathcal{F}(M)/G_\alpha$ be the Grassmann bundle of all tangent p_α-planes of M. $\mathcal{F}(M)$ and $\mathcal{G}^{p_\alpha}(M)$ are the associated fibred bundles. The distribution Δ^α on M defines a global cross-section:

$$\Delta^\alpha : x \in M \to \Delta^\alpha(x) = \Delta^\alpha_x \in \mathcal{G}^{p_\alpha}(M).$$

Let (U, φ) and (U, ψ) be the associated allowable charts on $\mathcal{F}(M)$ and $\mathcal{G}^{p_\alpha}(M)$, respectively. Thus we can define the morphism:

$$A_\alpha : \mathcal{F}(M) \to G^{p_\alpha}(n) = GL(n, R)/G_\alpha$$

by

$$A_\alpha(z) = [\varphi_x^{-1}(z)]^{-1} \cdot \psi_x^{-1}(\Delta^\alpha_x)$$

for $z \in \pi^{-1}(x)$ and $x \in U$, where φ_x (resp. ψ_x) is the restriction of φ (resp. ψ) to $\{x\} \times GL(n, R)$ (resp. $\{x\} \times G^{p_\alpha}(n)$). If $(\overline{U}, \overline{\varphi})$ and $(\overline{U}, \overline{\psi})$ are the other associated charts such that $x \in U \cap \overline{U}$ then:

$$[\overline{\varphi}_x^{-1}(z)]^{-1} \cdot \overline{\psi}_x^{-1}(\Delta^\alpha_x) = [a_{\overline{\varphi}\varphi}(x)\, \varphi_x^{-1}(z)]^{-1} (a_{\overline{\psi}\psi}(x) \cdot \psi_x^{-1})(\Delta^\alpha_x) =$$

$$= [\varphi_x^{-1}(z)]^{-1} [a_{\overline{\varphi}\varphi}(x)]^{-1} \cdot a_{\overline{\psi}\psi}(x)\, \psi_x^{-1}(\Delta^\alpha_x) = [\varphi_x^{-1}(z)]^{-1} \cdot \psi_x^{-1}(\Delta^\alpha_x)$$

where $a_{\overline{\varphi}\varphi}$ (resp. $a_{\overline{\psi}\psi}$) is the transition function of $\mathcal{F}(M)$ (resp. $\mathcal{G}^{p_\alpha}(M)$) corresponding to the charts (U, φ) and $(\overline{U}, \overline{\varphi})$, (resp. (U, ψ) and $(\overline{U}, \overline{\psi})$) and we have used the propriety $a_{\overline{\varphi}\varphi}(x) = a_{\overline{\psi}\psi}(x)$ which holds for associated charts.

It follows that A_α does not depend on the associated allowable charts. Since GL (n, R) acts transitively on $G^{p_\alpha}(n)$ and the isotropy group of e/G_α is G_α then $B_{G_\alpha}(M) = A_\alpha^{-1}(e/G_\alpha)$ is a principal fibre subbundle.

It follows that $B_{G_1}(M) \cap B_{G_2}(M)$ is a principal fibre subbundle if, and only if, for every $x \in M$, $\pi_1^{-1}(x) \cap \pi_2^{-1}(x) \neq \varnothing$, where π_1 and π_2 are the projections of $B_{G_1}(M)$ and $B_{G_2}(M)$, respectively.

Example 2. Let G (n) be the Grassmann manifold of all subspaces of R^n. It is well known [2], that G (n) is a compact manifold and $G^p(n)$ (the Grassmann manifold of p-subspaces of R^n) $p = 1, 2, \cdots, n$, is a connexe, open and closed submanifold. The group GL (n, R) acts differentiably on G (n), and $G^p(n)$ are its orbits.

Let $B_H(M)$ be a closed principal subbundle of a G-structure $B_G(M)$. If the homogeneous space G/H is isomorphic with an orbit of G with respect to the representation of G on G (n), then $B_H(M)$ is defined by a distribution Δ

on M (there exists a G-structure $B_{G_1}(M)$, as in Example 1, such that $B_H(M) = B_G \cap B_{G_1}$).

Indeed, since G is reducible to H there is a morphism $A_0 : B_G \to G/H$ (Lemma 1) such that $A_0(z \cdot g) = g^{-1} A_0(z)$ for all $z \in B_G$ and $g \in G$. Let us choose $G^p(n)$ such that $Gu_0 \subset G^p(n)$, and let p by the projection of $B_G(M)$. If $z_1 \in \pi^{-1}(x)$ (π is the projection of $\mathscr{F}(M)$) and $z_0 \in p^{-1}(x)$ then there exists $g \in GL(n, R)$ such that $z_1 = z_0 g$. We define the morphism $A : \mathscr{F}(M) \to G^p(n)$ by $A(z_1) = g^{-1} A_0(z_0)$.

Since $A_0(z_0) \in G^p(n)$, and $G^p(n)$ is an orbit of $GL(n, R)$, then $g^{-1} A_0(z_0) \subset G^p(n)$. Hence A takes its values in $G^p(n)$. If $z_1 = z_0 g$ with $z_0 \in p^{-1}(x)$, and $g \in G$, then $z_0 = z_0 g g^{-1}$ and hence

$$A(z_1) = g^{-1} A_0(z_0) = g^{-1} A_0(z_0 g g^{-1}) = g^{-1}(g g^{-1}) A_0(z_0) = g^{-1} A_0(z_0).$$

It follows that A is well defined. The statements of Lemma 1 are fulfilled and so $\mathscr{F}(M)$ is reducible to G_1. We obtain a global cross-section of the fibred bundle $\mathscr{F}(M)/G_1$. Let $B_{G_1}(M)$ be the reduced fibre bundle; it follows that $B_H(M) = B_G \cap B_{G_1}$.

REFERENCES

[1] D. BERNARD (1960) – *Thése*, «Ann. Inst. Fourier», *10*, 151–270.
[2] N. BOURBAKI (1967) – *Variétés différentielles et analytiques »–Résultats*, Hermann, Paris.
[3] S. KOBAYASHI and K. NOMIZU (1965) – *Foundations of Differential Geometry*, Interscience, New York, vol. I.