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Gary D. Jones

Oscillation properties of $y'' + p(x)y = f(x)$

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Equazioni differenziali ordinarie. — *Oscillation properties of* $y'' + p(x) y = f(x)$. Nota di GARY D. JONES, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà alcuni teoremi oscillatori per le soluzioni dell'equazione $(t)\, y'' + p(x) y = f(x)$ nel caso $f > 0$. È data anche una condizione sufficiente per l'esistenza di soluzioni oscillatorie della (k) nel caso che $f(x)$ cambi di segno.

1. INTRODUCTION

The purpose of this paper is to study the equation

$$y'' + p(x) y = f(x), \tag{1}$$

which has recently been studied by Keener [3] and Leighton and Skidmore [5]. The method used here will be that of relating (1) to a third order linear homogeneous equation as used earlier by Svec [6]. Some of the results obtained in this way will generalize results in [3] and [5]. Others will be new. It will be assumed that p and f are in $C^1(0, +\infty)$.

To say that a solution y of (1) is oscillatory we will mean that it has zeros for arbitrarily large x. To say that (1) is oscillatory we will mean that there is an oscillatory solution of (1).

2. A THIRD ORDER EQUATION

If $f(x) > 0$, it is easy to see that every solution of (1) is a solution of

$$(y''/f)' + (p/f) y' + (p/f)' y = 0 . \tag{2}$$

Also every solution of (2) is a solution of

$$y'' + py = 0 \tag{3}$$

or a nonzero multiple of a solution of (1).

We will be interested in equations (1) such that (2) is $\mathbf{C_I}$ or C_{II} as defined by Hanan [1].

DEFINITION. *Equation* (2) *is* C_I *if any solution for which* $y(a) = y'(a) = 0$, $y''(a) > 0$ *is positive on* $[0, a)$. *It is said to be* C_{II} *if any solution for which* $y(a) = y'(a) = 0$, $y''(a) > 0$ *is positive on* $(a, +\infty)$.

Assuming $r(x) > 0$, we now give three theorems which are known for $r = 1$, for

$$(ry'')' + py' + qy = 0 \tag{I}$$

(*) Nella seduta del 14 novembre 1974.

and

$$\text{(II)} \qquad (rz')'' + pz' + (p' - q) z = 0 .$$

THEOREM 1. *Equation* (I) *is* C_I (C_{II}) *if* $r' \geq 0$ *and* $2q - p' \geq 0$ $(r' \leq 0$ *and* $2q - p' \leq 0)$ *where* $r' + (2q - p')$ *can equal zero only at isolated points.*

Proof. Let $y(x)$ be a solution of (I) such that $y(a) = y'(a) = 0$, $y''(a) = 1$. Suppose (I) is not C_I. Let b be the first zero of $y(x)$ to the left of a. Multiplying (I) by y and integrating from b to a we have

$$2yry'' - ry'^2 + py^2 \,|_b^a = \int_b^a (p' - 2q) y^2 - \int_b^a r' y'^2 .$$

Thus $r(b) y'^2(b) < 0$ which is a contradiction.

In the same way with $r' \leq 0$ and $2q - p' \leq 0$, we can show (I) is C_{II}.

COROLLARY 1. *Equation* (2) *is* C_I (C_{II}) *if* $f' \leq 0$ *and* $(p/f)' \geq 0$ $(f' \geq 0$ *and* $(p/f)' \leq 0)$ *with only isolated zeros of* $(1/f)' + (p/f)'$.

THEOREM 2 [4]. *If* $(rz')' + pz = 0$ *is nonoscillatory and* $q > 0$ $(q < 0)$ *then* (I) *is* C_I (C_{II}).

COROLLARY 2. *If* $(z'/f)' + (p/f) z = 0$ *is nonoscillatory and* $(p/f)' > 0$ $((p/f)' < 0)$ *then* (2) *is* C_I (C_{II}).

THEOREM 3. *Equation* (I) *is* C_I (C_{II}) *if and only if* (II) *is* C_{II} (C_I).

Proof. Proceeding in the same manner as Hanan for the case $r \equiv 1$ [1], multiply (I) by a solution z of (II) and (II) by a solution y of (I). Adding, we obtain

$$0 = y(rz')'' + z(ry'')' + (pz)' y + zpy'$$

Suppose (II) is C_{II} but (I) is not C_I. Suppose $y(b) = y'(b) = 0$, $y''(b) = 1$ and that $y(a) = 0$ for $a < b$. Let z be the solution of (II) defined by $z(a) = z'(a) = 0$, $(rz')'(a) = 1$. Integrating the above expression from a to b we have

$$0 = y(rz')' - y' rz' + zry'' + pzy \,|_a^b .$$

Thus we have

$$0 = z(b) r(b) y''(b),$$

which is a contradiction. Similar arguments complete the proof.

From Corollary 1 we see that the monotone properties of f and p assumed by Keener [3] and Leighton and Skidmore [5] force (2) to be C_I or C_{II}. We will obtain some of their conclusions with the hypothesis that (2) be C_I or C_{II} thus generalizing their corresponding results.

3. NONOSCILLATORY SOLUTIONS

In view of Corollary 1, the following theorems supplement results of Keener [3] and Leighton and Skidmore [5].

THEOREM 4. *If* $f > 0$ *and* (2) *is* C_{II}, *then there are three linearly independent nonoscillatory solutions of* (2) *that are solutions of* (1).

Proof. If (1) is oscillatory then (3) must be oscillatory [3]. Since (2) is C_{II}, it has three linearly independent nonoscillatory solutions z_1, z_2, and z_3 [2]. Since z_i cannot be a solution of (3), then $k_i z_i$ is a solution of (1) for nonzero constant k_i.

In general the conclusion of Theorem 4 is not true when (2) is C_I. However, proceeding as in the proof of the above theorem, using the fact that if (2) is C_I it has a solution with no zeros, we obtain the following theorem.

THEOREM 5. *If* $f > 0$, (3) *is oscillatory and* (2) *is* C_I *then there is a solution of* (1) *with no zeros.*

THEOREM 6. *If* $f > 0$, $f' \leq 0$, $(p/f)' > c > 0$, *and* (1) *is oscillatory, then there is a unique nonoscillatory solution of* (1).

Proof. By Corollary 1 equation (2) is C_I. Thus, by Theorem 3

$$(z'/f)'' + (p/f)z' = 0 \tag{4}$$

is C_{II}. Using the methods of [2] (4) has a basis consisting of 2 oscillatory solutions and one nonoscillatory solution.

Every solution z of (4) satisfies

$$F[z(x)] \equiv z(z'/f)' - z'^2/2f + pz^2/2f = \int_a^x (1/f)' z'^2/2 + \int_a^x (p/f)' z^2/2 + F[z(a)].$$

Clearly $F[z(x)]$ is a nondecreasing function of x. If z is a nontrivial oscillatory solution then $F[z(x)] < 0$ since it is negative at the zeros of z. Now

$$\frac{1}{2}\int_a^x z^2 \leq \frac{1}{c}\int_a^x (p/f)' z^2/2 \leq \frac{1}{c}\left[\int_a^x (p/f)' z^2/2\right] +$$

$$+ \int_a^x (1/f)' z'^2/2 = \{F[z(x)] - F[z(a)]\}/c \leq -F[z(a)]/c.$$

Thus z is square integrable. By the Minkowski inequality $d + z$ is not square integrable for any constant $d \neq 0$. Since d is a solution of (4) it follows that every oscillatory solution of (4) must be a linear combination of the two oscillatory solutions of the basis given above. Again by the methods of [2] it follows that every nonoscillatory solution of (2) must be a multiple of the one given by Theorem 5. Thus (1) has a unique nonoscillatory solution.

4. Oscillation properties

In view of the remarks in Section 2 we obtain several properties of oscillatory solutions of (1) by considering (2). Some are listed below. Theorem 7 below generalizes Keener's Theorem 7 [3] and Leighton and Skidmore's Theorem 2.3 [5]. Theorem 10 generalizes Theorem 2.4 in [5].

THEOREM 7. *If* $f > 0$ *and* (2) *is* C_I $[C_{II}]$ *and* y *is a solution of* (1) *such that* $y(a) = y'(a) = 0$, *then* $y(x) > 0$ *for* $x \in [0, a)$ $[x \in (a, +\infty)]$.

THEOREM 8. *If* $f > 0$ *and* (2) *is* C_I *or* C_{II} *then two different solutions of* (1) *cannot have two common zeros.*

Proof. This follows from [7, 4.7, p. 153].

THEOREM 9. *Let* $f > 0$ *and* (2) *be* $C_I (C_{II})$. *If* u *and* v *are different solutions of* (1) *such that* $u(\alpha) = v(\alpha) = 0$, *then the zeros of* u *and* v *separate in* $(\alpha, +\infty)$ [*in* $(0, \alpha)$].

Proof. This follows from [7, 4.8, p. 153].

THEOREM 10. *Let* $f > 0$ *and* (2) *be* C_I. *If* (3) *is oscillatory every solution of* (1) *which has a zero is oscillatory.*

Proof. This follows from [7, 4.10, p. 154].

5. A condition for oscillation

In this section we give a sufficient condition for oscillation of (1) where $f(x)$ can have zeros.

THEOREM 11. *If* $p' > 0, p > 0$ *and* $\int_0^\infty |f'| < \infty$ *then* (1) *has an oscillatory solution.*

Proof. Since

$$f(x) = \left[\int_0^\infty |f'| - n(x)\right] - \left[\int_0^\infty |f'| - q(x)\right] + f(0)$$

where $q(x)$ and $n(x)$ are the positive and negative variation of f on $[0, x]$, it is clear that f can be written as $f = f_1 - f_2$ where f_i are positive nonincreasing functions in $C^1[0, \infty)$.

Now

$$(y''/f_i)' + (p/f_i)\, y' + (p/f_i)'\, y = 0 \tag{5}$$

is C_I for $i = 1, 2$. Let z_i be the nonoscillatory solutions for

$$y'' + py = f_i \tag{6}$$

whose existence was shown above. By a result of Keener [4], z_i must eventually be positive. By Theorem 10 if z_i has a zero it is oscillatory. Thus it is always positive. Let v be an oscillatory solution of (3) and $a, b > 0$ such that $v(a) < 0$ and $v(b) > 0$. Choose a constant k so that $kv(a) + z_1(a) < 0$ and $-kv(b) + z_2(b) < 0$. Thus by Theorem 10 the functions $kv + z_1$ and $-kv + z_2$ are oscillatory solutions of (6) for $i = 1$ and 2 respectively. The function $kv + z_1 - z_2$ is a solution of (1). If $kv + z_1 - z_2$ is eventually positive, then $(kv + z_1 - z_2) + z_2$ is eventually positive, which is a contradiction. If $kv + z_1 - z_2$ is eventually negative, then $(kv + z_1 - z_2) - z_1$ is eventually negative which is also a contradiction. Thus $kv + z_1 - z_2$ is an oscillatory solution (1).

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