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Periodic Solutions of Certain fourth order differential equations

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Equazioni differenziali. — *Periodic Solutions of Certain fourth order differential equations.* Nota [(*)] di H. O. TEJUMOLA, presentata dal Socio G. SANSONE.

RIASSUNTO. — L'Autore sotto opportune ipotesi prova l'esistenza di una soluzione periodica dell'equazione

$$x^{(4)} + a_1 \dddot{x} + g(\dot{x})\ddot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}) = p(t),$$

dove $p(t)$ è una funzione periodica nota.

1. Consider the real differential equation

$$x^{(iv)} + a_1 \dddot{x} + g(\dot{x})\ddot{x} + a_3 \dot{x} + h(x) = p(t), \tag{1.1}$$

where $a_1 > 0$, $a_3 > 0$ are constants, g, h and p are continuous functions of the arguments shown in (1.1) and the function h is bounded, that is

$$|h(x)| \leq H \qquad \text{(H a constant) for all } x.$$

The equation (1.1) has been investigated by a number of authors for the boundedness of solutions. Ezeilo [1], for example, in generalizing an earlier result of Reissig [5] for the special case $g(\dot{x}) = a_2$ (a_2 constant), showed that all solutions of (1.1) are ultimately bounded if $h(x) \operatorname{sgn} x > 0$ ($|x| \geq x_0$), $P(t) \equiv \int_0^t p(\tau)\, d\tau$ is bounded for all $t \geq 0$ and if there are constants $A_3 \geq 0$ and A_2 satisfying $A_2 > a_1^{-1} a_3$ such that

$$G(y) \operatorname{sgn} y \geq A_2 |y| - A_3 \qquad \text{for all } y, \; G(y) \equiv \int_0^y g(s)\, ds. \tag{1.2}$$

This result was further extended by the present Author [6] in a recent paper.

When p is a periodic function there seem to be fewer results on the existence of periodic solutions of (1.1) under the same or similar conditions on g as in [1]. The only well known result in this direction is that of Reissig [4] which, when specialized to $n = 4$, concerns the case $g(\dot{x}) = a_2$ (a_2 constant) in (1.1) with $a_2 > a_1^{-1} a_3$; the later inequality being the analogue of (1.2) in this case.

The object of this note is to prove an existence result for (1.1) under conditions similar to those of Ezeilo above. We shall in fact be concerned with the more general equation

$$x^{(iv)} + a_1 \dddot{x} + g(\dot{x})\ddot{x} + a_3 \dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}) = p(t), \tag{1.3}$$

(*) Pervenuta all'Accademia il 27 agosto 1974.

where h is a continuous function depending on all of the arguments shown and h is bounded, that is

$$(1.4) \qquad |h(x, y, z, u)| \leq H \qquad (H \text{ a constant})$$

for all x, y, z and u.

The following is our main result:

THEOREM. *Let p be periodic in t with period ω and let h satisfy* (1.4) *and*

$$(1.5) \qquad h(x, y, z, u) \operatorname{sgn} x > 0 \qquad (|x| \geq x_0).$$

Suppose further that there are constants $A_0 \geq 0$, $A_1 \geq 0$ *and* $a_2 > 0$ *satisfying*

$$(1.6) \qquad a_2 > a_1^{-1} a_3$$

such that

$$(1.7) \qquad |P(t)| \equiv \left| \int_0^t p(\tau)\, d\tau \right| \leq A_0 \qquad (t \geq 0),$$

$$(1.8) \qquad G(y) \operatorname{sgn} y \geq a_2 |y| - A_1 \qquad \textit{for all } y,\ G(y) \equiv \int_0^y g(s)\, ds.$$

Then the equation (1.3) *admits at least one periodic solution with period* ω.

2. Some Preliminaries

The procedure for the proof of the Theorem is essentially the same as in [2]. Consider the parameter (μ)–dependent equation

$$(2.1) \qquad \begin{aligned} &x^{(iv)} + a_1 \dddot{x} + \{(1 - \mu) a_2 + \mu g(\dot{x})\} \ddot{x} + a_3 \dot{x} + \\ &+ (1 - \mu) a_4 x + \mu h(x, \dot{x}, \ddot{x}, \dddot{x}) = \mu p(t), \qquad 0 \leq \mu \leq 1, \end{aligned}$$

which reduces to the original equation (1.3) when $\mu = 1$ and to the linear equation

$$(2.2) \qquad x^{(iv)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0$$

when $\mu = 0$. Here a_4 is a constant to be fixed such that the linear equation (2.2) is asymptotically stable. Following Reissig [4] define, for some constant S, an auxilliary equation as follows:

$$(2.3) \qquad x^{(iv)} + a_1 \dddot{x} + g_\mu(\dot{x}) \ddot{x} + a_3 \dot{x} + h_\mu(x, \dot{x}, \ddot{x}, \dddot{x}) = \mu p(t), \qquad 0 \leq \mu \leq 1,$$

where

$$(2.4) \qquad \begin{cases} g_\mu(\dot{x}) = (1 - \mu) a_2 + \mu g(\dot{x}) \\ h_\mu(x, \dot{x}, \ddot{x}, \dddot{x}) = \begin{cases} (1 - \mu) a_4 x + \mu h(x, \dot{x}, \ddot{x}, \dddot{x}), & \text{if } |x| \leq S, \\ (1 - \mu) a_4 S \operatorname{sgn} x + \mu h(x, \dot{x}, \ddot{x}, \dddot{x}), & \text{if } |x| \geq S. \end{cases} \end{cases}$$

Since h_μ is bounded, indeed by (1.4),

$$(2.5) \qquad |h_\mu(x, y, z, u)| \leq a_4 S + H \qquad \text{for all} \quad x, y, z, u,$$

the equation (2.3), in contrast with (2.1), is amenable to the techniques developed in [3] for boundedness of solutions. In fact it suffices here to show that (I) all solutions of (2.3) are ultimately bounded, with bound independent of solutions and of μ $(0 \leq \mu \leq 1)$, and (II) that for a suitable choice of S and a_4, every solution $x(t)$ of (2.3) ultimately satisfies $|x(t)| \leq S$. For since the equation (2.3) reduces to (2.1) when $|x| \leq S$, an application of the Leray-Schauder fixed point technique to (2.3) would show, by (I), that the equation (2.1) admits a periodic solution with period ω and this would imply the existence of an ω-periodic solution of the original equation (1.3).

In what follows two sets of constants will be used. The letters $d_1, d_2, \cdots$ denote finite positive constants whose magnitudes depend only on $a_1, a_2, a_3, a_4, x_0, A_0$ and A_1 but are independent of μ and S. Each d_i, $i = 1, 2, \cdots$ retains the same identity throughout. The second set of constants are $\delta, \delta_1, \delta_2, \cdots$. Each δ, with or without subscript, denotes a constant whose magnitude depends only on $a_1, a_2, a_3, a_4, x_0, A_0$, and A_1 as well as on S and g, but definitely not on μ. Each of the numbered δ's retains a fixed identity throughout but the unnumbered ones are not necessarily the same each time they occur. To emphasize the dependence of a δ on another constant, say η, we shall write $\delta(\eta)$.

Let $\bar{G}(y) = \int_0^y G(s)\, ds$ and define a function $\bar{G}^*$ on $[0, \infty)$ by

$$\bar{G}^*(y) = \max_{|\xi| \leq y} |\bar{G}(\xi)|.$$

Since G is continuous, it is clear that $\bar{G}^*$ is a non-decreasing continuous function on $[0, \infty)$ such that $\bar{G}^*(0) = 0$ (since $\bar{G}(0) = 0$) and $|\bar{G}(y)| \leq \bar{G}^*(|y|)$ for all y. Since $|y| = \sqrt{(y^2)}$ we may now define a function G^* on $[0, \infty)$ by setting

$$G^*(y^2) = \bar{G}^*(|y|).$$

Evidently G^* is also a continuous non-decreasing function $[0, \infty)$ such that $G^*(0) = 0$ and

$$|\bar{G}(y)| \leq G^*(y^2) \qquad \text{for all} \quad y.$$

As a first step towards the verification of (I) and (II) above, we shall show that there are constants $d_0, d_1, \delta_0(\eta)$ and a continuous function $\Delta(y)$ such that every solution $x(t)$ of (2.3) ultimately satisfies

$$(2.6) \qquad |x(t)| \leq x_0 + 2 d_0 \{1 + \delta_0(\eta) + \Delta(\eta) + G^*(\Delta(\eta))\},$$

$$(2.7) \qquad \max(|\dot{x}(t)|, |\ddot{x}(t)|, |\dddot{x}(t)|) \leq d_1 \{1 + \delta_0(\eta) + \Delta(\eta) + G^*(\Delta(\eta))\},$$

where x_0 is the constant in (1.5) and

$$(2.8) \qquad \eta = a_1 a_3^{-1}(a_4 S + H) + A_0 + 1.$$

3. A FUNCTION $V(x, y, z, u)$

The main tool for the proof (2.7) is a slightly modified form of the function V used in [3], which in terms of our present notations, is given by

$$V = V_0 - 2\eta V_1 - \eta V_2 , \tag{3.1}$$

where

$$\left\{\begin{aligned} & V_0 = \int_0^y G_\mu(s)\, ds - yz + \frac{1}{2}(u^2 + a_1 a_3^{-1} z^2), \quad G_\mu(y) \equiv \int_0^y g_\mu(s)\, ds , \\ & V_1 = \begin{cases} u \operatorname{sgn} z , & \text{if } |z| \geq |u| , \\ z \operatorname{sgn} u , & \text{if } |u| \geq |z| , \end{cases} \\ & V_2 = \begin{cases} y \operatorname{sgn} u , & \text{if } |u| \geq |y| , \\ u \operatorname{sgn} y , & \text{if } |y| \geq |u| . \end{cases} \end{aligned}\right. \tag{3.2}$$

First, we show that V satisfies

$$\begin{aligned} -d_3(\eta^2 + 1) + d_2(y^2 + z^2 + u^2) \leq V \leq d_4(y^2 + z^2 + u^2) + \\ + G^*(y^2 + z^2 + u^2) + d_5(\eta^2 + 1) \end{aligned} \tag{3.3}$$

for some d_2, d_3, d_4 and d_5. Indeed

$$V_0 = \int_0^y \{G_\mu(s) - a_1^{-1} a_3 s\}\, ds + \frac{1}{2} u^2 + \frac{1}{2}(a_1^{1/2} a_3^{-1/2} z - a_1^{-1/2} a_3^{1/2} y)^2 \geq$$

$$\geq \frac{1}{2}(a_2 - a_1^{-1} a_3) y^2 + \frac{1}{2} u^2 + \frac{1}{2}(a_1^{1/2} a_3^{-1/2} z - a_1^{-1/2} a_3^{1/2} y)^2 - A_1 |y| ,$$

by (1.8) and the fact that $0 \leq \mu \leq 1$. Thus

$$V_0 \geq d_6(y^2 + z^2 + u^2) - d_7$$

for some d_6 and d_7, and by (3.1), the last inequality implies the left-hand inequality in (3.3), since $|V_1| \leq |u|$ and $|V_2| \leq |u|$. Observe next from the definitions of G_μ and G^* that

$$\int_0^y G_\mu(s)\, ds \leq a_2 y^2 + G^*(y^2) \qquad \text{for all } y,$$

so that, in view of (3.2)

$$V_0 \leq G^*(y^2 + z^2 + u^2) + a_2 y^2 + \frac{1}{2}(y^2 + z^2) + \frac{1}{2}(u^2 + a_1 a_3^{-1} z^2), \tag{3.4}$$

since G^* is non-decreasing. The right-hand inequality in (3.3) is implied by (3.4) since $|V_1| \leq |u|$ and $|V_2| \leq |u|$.

Consider (2.4) in the system form

$$\dot{x} = y \quad , \quad \dot{y} = u - a_1 y \quad , \quad \dot{z} = -a_3 y - h_\mu(x, y, v, w) \tag{3.5}$$
$$\dot{u} = z - G_\mu(y) + \mu P(t),$$

where $v = u - a_1 y$, $w = z - G_\mu(y) + \mu P(t)$. For any solution $(x, y, z, u) \equiv \equiv (x(t), y(t), z(t)), u(t))$ of (3.5), let $\dot{V}^*$ have its usual meaning. We shall show that

$$\dot{V}^* \leq -1 \qquad \text{if} \quad y^2 + z^2 + u^2 \geq \delta_0^2(\eta), \tag{3.6}$$

where $\delta_0(\eta)$ is a continuous function of η. Indeed a simple calculation from (3.1) and (3.5) will show that

$$\dot{V}^* \leq -a_1 y \{G_\mu(y) - a_1^{-1} a_3 y\} + (a_4 S + H)(|y| + a_1 a_3^{-1} |z|) +$$
$$+ A_0 |u| + M_1 + M_2,$$

where

$$M_1 \leq \begin{cases} -2\eta |z| + \delta(\eta)(|G_\mu(y)| + 1), & \text{if } |z| \geq |u|, \\ \delta(\eta)(|y| + 1), & \text{if } |u| \geq |z|, \end{cases}$$

$$M_2 \leq \begin{cases} -\eta |u| + \delta(\eta)|y|, & \text{if } |u| \geq |y|, \\ \eta |z| + \delta(\eta)(|G_\mu(y)| + 1), & \text{if } |y| \geq |u|. \end{cases}$$

Since η is given by (2.8), we therefore have that

$$\dot{V}^* \leq \begin{cases} -a_1 y \{G_\mu(y) - a_1^{-1} a_3 y\} - \{a_1 a_3^{-1}(a_4 S + H) + 2A_0 + 2\} |z| - \\ \quad - \{2 a_1 a_3^{-1}(a_4 S + H) + A_0 + 2\} |u| + \delta(\eta)(|G_\mu(y)| + |y| + 1), \\ \qquad\qquad \text{if } |z| \geq |u| \geq |y|, \\ -a_1 y \{G_\mu(y) - a_1^{-1} a_3 y\} - |z| + \delta(\eta)(|G_\mu(y)| + |y| + 1), \\ \qquad\qquad \text{if } |z| \geq |u| \text{ and } |y| \geq |u|, \\ -a_1 y \{G_\mu(y) - a_1^{-1} a_3 y\} - |u| + \delta(\eta)(|y| + 1), \\ \qquad\qquad \text{if } |u| \geq |z| \text{ and } |u| \geq |y|, \\ -a_1 y \{G_\mu(y) - a_1^{-1} a_3 y\} + \delta(\eta)(|G_\mu(y)| + |y| + 1), \\ \qquad\qquad \text{if } |y| \geq |u| \geq |z|. \end{cases} \tag{3.7}$$

The inequality (3.7) is the analogue of (4.3) of [3] and we have, corresponding to (4.4) of [3], that $\dot{V}^*$ satisfies

$$\dot{V}^* \leq -a_1 y \{G_\mu(y) - a_1^{-1} a_3 y\} + \delta_1(\eta)(|G_\mu(y)| + |y|) + \delta(\eta) \tag{3.8}$$

always, for some $\delta_1(\eta)$, $\delta(\eta)$. Observe from (1.8) that

$$\{G_\mu(y) - a_1^{-1} a_3 y\} \operatorname{sgn} y \geq (a_2 - a_1^{-1} a_3) |y| - A_1 \qquad \text{for all } y,$$

so that in view of (1.6),

$$\{G_\mu(y) - a_1^{-1} a_3 y\} \operatorname{sgn} y \to +\infty \qquad \text{as } |y| \to \infty .$$

Thus the various arguments employed in [3 ; § 4] apply here; the results (3.7) and (3.8) can be used in precisely the same way as (4.3) and (4.4) of [3] to yield the result

$$\dot{V}^* \leq -1 \qquad \text{if } y^2 + z^2 + u^2 \geq \delta_2^2(\eta)$$

for some $\delta_2(\eta)$. This verifies (3.6).

4. Verification of (2.6) and (2.7).

We start with (2.7), the actual proof of which depends on the results (3.3) and (3.6). Observe that for any solution $(x(t), y(t), z(t), u(t))$ of (3.5),

$$y^2(t_0) + z^2(t_0) + u^2(t_0) < \delta_0^2(\eta) \tag{4.1}$$

for some $t_0 \geq 0$. For otherwise we would have $\dot{V}^* \leq -1$, and so $V \to -\infty$ as $t \to +\infty$, contrary to the estimate $V \geq -d_3(\eta^2 + 1)$ in (3.3). Next we show that

$$y^2(t) + z^2(t) + u^2(t) \leq \delta_3^2(\eta) \qquad \text{for all } t \geq t_0, \tag{4.2}$$

where

$$\delta_3^2(\eta) = d_2^{-1}[(d_3 + d_5)(\eta^2 + 1) + (d_2 + d_4)\delta_0^2(\eta) + G^*(\delta_0^2(\eta))] . \tag{4.3}$$

Since $\delta_3(\eta) > \delta_0(\eta)$, if (4.2) were false, there would exist $T_0 > t_0$ such that

$$y^2(T_0) + z^2(T_0) + u^2(T_0) > \delta_3^2(\eta)$$

and this, since $y^2(t) + z^2(t) + u^2(t)$ is continuous, would imply the existence of t_1, t_2 with $t_2 > t_1 > t_0$ such that

$$y^2(t_2) + z^2(t_2) + u^2(t_2) = \delta_3^2(\eta) \quad , \quad y^2(t_1) + z^2(t_1) + u^2(t_1) = \delta_0^2(\eta) \tag{4.4}$$

and

$$y^2(t) + z^2(t) + u^2(t) \geq \delta_0^2(\eta) \qquad (t_1 \leq t \leq t_2) . \tag{4.5}$$

But (4.5) and (3.6) imply that $V(t_2) < V(t_1)$ whilst (4.4) and (3.3) give that

$$\begin{aligned} V(t_2) &\geq d_2\,\delta_3^2(\eta) - d_3(\eta^2 + 1) > \\ &> d_4\,\delta_0^2(\eta) + G^*(\delta_0^2(\eta)) + d_5(\eta^2 + 1) \geq \\ &\geq V(t_1) , \end{aligned}$$

which is a contradiction. Therefore (4.2) holds.

The result (4.2) obviously implies that

$$\max(|y(t)|, |z(t)|, |u(t)|) \leq \delta_3(\eta) \qquad \text{for all} \quad t \geq t_0 .$$

Since each $\delta(\eta)$, whether numbered or not, which featured in the preceding arguments, is a continuous function of η, it follows that the result (2.7) is, in view of (3.5), implied by (4.2).

We turn now to (2.6). The details of the proof here are as in [6; § 3] (see also [2; § 4]) and we shall merely sketch the outline. Let $(x, y, z, u) = (x(t), y(t), z(t), u(t))$ be any solution of (3.5) and define the function $\psi = \psi(t)$ by

$$\psi = z + a_3 x. \tag{4.6}$$

Then, in view of (3.5), (2.4) and (1.5),

$$\dot{\psi} < 0 \qquad \text{if} \quad x \geq x_0 . \tag{4.7}$$

We also have by (2.7) that

$$|\psi - a_3 x| \leq d_1\{1 + \delta_0(\eta) + \Delta(\eta) + G^*(\Delta(\eta))\}, \qquad t \geq T_0 \tag{4.8}$$

for some T_0. If therefore suffices to show that

$$|\psi(t)| \leq a_3 x_0 + d_1\{1 + \delta_0(\eta) + \Delta(\eta) + G^*(\Delta(\eta))\}, \qquad t \geq T_1 \tag{4.9}$$

for some $T_1 \geq T_0$. For, by (4.6) and (4.8), this implies that

$$a_3 |x(t)| \leq a_3 x_0 + 2 d_1\{1 + \delta_0(\eta) + \Delta(\eta) + G^*(\Delta(\eta))\}, \qquad t \geq T_1 ,$$

which is (2.6) with $d_0 = a_3^{-1} d_1$. The actual proof of (4.9) follows as in that of the corresponding result (3.4) of [6] (or (4.8) of [2]) using (4.6), (4.7) and (4.8) as required; further details will be omitted here.

5. Completion of the proof of the Theorem

In view of our earlier remarks it remains now to show that there is a suitable choice of the constants $a_4 > 0$ and $S > 0$ such that (I) the linear equation

$$x^{(iv)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0 \tag{5.1}$$

is asymptotically stable and (II), every solution $x(t)$ of (2.3) ultimately satisfies $|x(t)| \leq S$.

First we verify (II). Define a function $\chi : R^+ \to R^+$ by

$$\chi(y) = x_0 + 2 d_0\{1 + \delta_0(y) + \Delta(y) + G^*(\Delta(y))$$

and let

$$\bar{\chi}(r) = \max_{|y| \leq r} \chi(y) .$$

Set $d_6 = a_1 a_3^{-1}$, $d_7 = d_6 \mathrm{H} + \mathrm{A}_0 + 1$. In view of (2.6) and (2.8), the condition (II) would be met by a choice of the constants a_4, S satisfying

$$\mathrm{S} \geq \chi(d_6 a_4 \mathrm{S} + d_7). \tag{5.2}$$

It is in fact easy to check that (5.2) is satisfied by, for example,

$$\mathrm{S} \equiv \bar{\chi}(d_7 + 1) = d_8 \quad , \quad a_4 = d_6^{-1}(d_8 + 2)^{-n} \tag{5.3}$$

for any real number $n \geq 1$.

Turning to (I), observe that the Routh–Hurwitz stability criteria

$$a_i > 0 \,, \quad i = 1, 2, 3, 4 \,, \qquad (a_1 a_2 - a_3) a_3 - a_1^2 a_4 > 0 \tag{5.4}$$

are sufficient for the asymptotic stability of (5.1). In view of (1.6), (5.4) would be met by the value a_4 in (5.3) if n is chosen large enough to ensure that

$$(\mathrm{d}_8 + 2)^{-n} < a_1^{-1}(a_1 a_2 - a_3).$$

This completes the proof of the Theorem.

6. Remarks

The Theorem extends readily to equations of the form

$$x^{(\mathrm{iv})} + f(\ddot{x})\,\dddot{x} + g(\dot{x})\,\ddot{x} + a_3 \dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}) = p(t), \tag{6.1}$$

in which the constant a_1 in (1.3) is replaced by a continuous function f satisfying

$$\mathrm{F}(v) - a_1 v = \mathrm{o}(1) \quad \text{as} \quad |v| \to \infty \quad , \quad \mathrm{F}(v) = \int_0^v f(s)\,\mathrm{d}s \tag{6.2}$$

but with the constants a_1, a_3 and the functions g, h and p as before.

The proof of the theorem in this case is exactly the same as for the equation (1.3) except that (2.3) has to be replaced by

$$x^{(\mathrm{iv})} + f_\mu(\ddot{x})\,\dddot{x} + g_\mu(\dot{x})\,\ddot{x} + a_3 \dot{x} + h_\mu(x, \dot{x}, \ddot{x}, \dddot{x}) = \mu p(t)$$

with g_μ and h_μ as in (2.3) and

$$f_\mu(\ddot{x}) = (1 - \mu)\, a_2 + \mu f(\ddot{x}),$$

and the system (3.5) by

$$\begin{aligned} &\dot{x} = y \quad , \quad \dot{y} = u - a_1 y \quad , \quad \dot{z} = -a_3 y - h_\mu(x, y, v, w), \\ &\dot{u} = z - \mathrm{G}_\mu(y) - \{\mathrm{F}_\mu(v) - a_1 v\} + \mu \mathrm{P}(t) \quad , \quad \mathrm{F}_\mu(v) \equiv \int_0^v f_\mu(s)\,\mathrm{d}s \end{aligned} \tag{6.3}$$

with $v = u - a_1 y$, $w = z - G_\mu(y) - F_\mu(v) + \mu P(t)$. Also, because of the term $F_\mu(v) - a_1 v$ which features in (6.3) but is absent in (3.5) a new choice of η has to be made. Indeed, since

$$| F_\mu(v) - a_1 v | \leq B \qquad \text{for all} \quad v$$

for some constant $B \geq 0$, the choice

$$\eta = a_1 a_3^{-1} (a_4 S + H) + A_0 + B + 1$$

will suffice.

REFERENCES

[1] J. O. C. EZELIO (1972) – « J. Lond. Math. Soc. » (2), *5*, 376–384.

[2] J. O. C. EZEILO (1973) – « Atti Acad. Naz. Lincei, Rend. Cl. Sci. fis. mat. nat. », (8) *54*, 34–41.

[3] J. O. C. EZEILO and H. O. TEJUMOLA – *Boundedness Theorems for certain third order differential equations*, to appear in « Atti Accad. Naz. Lincei, Rend. Cl. Sci. fis. mat. nat. ».

[4] R. REISSIG (1970) – « Ann. Mat. Pura Appl. » (iv), *85*, 235–240.

[5] R. REISSIG (1965) – « Monatsb. Deutsch Akad. Wiss Berlin », *7*, 251–256.

[6] H. O. TEJUMOLA – *An extension of a boundedness theorem of Ezeilo*, to appear in « Bolletino U.M.I. ».