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Some fixed point problems

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Topologia. — *Some fixed point problems.* Nota (*) di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota sono elencati dieci problemi, ritenuti dall'Autore non risolti, sui punti fissi di spazi metrici, normati e localmente convessi.

Many contributions have recently been made to fixed point theory in complete metric, normed linear and locally convex spaces. An extensive bibliography can be found in [9]. In this note we wish to draw the reader's attention to several problems which are, to the best of our knowledge, still open. Although these problems are easy to state, it seems that a new idea is necessary in order to solve each one of them.

Let D be a non-empty subset of a Banach space $(E, \| \ \|)$. A mapping $f: D \to E$ is said to be non-expansive if $\|f(x) - f(y)\| \leq \|x - y\|$ for all x and y in D. A subset of E is said to have the fixed point property for non-expansive mappings of every non-expansive self-mapping of it has a fixed point.

PROBLEM 1. *Does every weakly compact convex subset of a Banach space possess the fixed point property for non-expansive mappings?*

We cannot apply Tychnoff's fixed point theorem [34, p. 770] and obtain immediately an affirmative answer because a non-expansive mapping is not necessarily weakly continuous. Simple examples [19] show that when " weakly compact " is replaced by " bounded and closed " or " convex " is omitted, or " non-expansive " is replaced by " Lipschitzian " (with a Lipschitz constant $k > 1$), the answer is in the negative.

The best positive result to date is still that of Kirk [19, p. 1004]. He showed that the answer is in the affirmative if we postulate that the subset in question has normal structure. The same result follows from Browder's proof of a less general theorem [5, p. 1041]. Unfortunately, not all weakly compact convex subsets of Banach spaces possess normal structure [2, p. 439]. Another sufficient condition which was mentioned in [30, p, 254] is in fact equivalent to normal structure by the results of [4] and [22]. Let E be a uniformly convex Banach space. There exists $t > 1$ with the following property: A weakly compact convex subset of a Banach space whose Banach-Mazur distance coefficient with respect to E is smaller than t has the fixed point property for non-expansive mappings. This follows from [14, Theorem 1].

In order to state a related problem we recall that a subset of a Banach space is said to have the common fixed point property for non-expansive map-

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pings if every commuting family of non-expansive self-mappings of it has a common fixed point.

PROBLEM 2. *Suppose that* C, *a weakly compact convex subset of a Banach space, possesses the fixed point property for non-expansive mappings. Does it also have the common fixed point property for non-expansive mappings?*

The most general result at present requires that C may have, in addition to the fixed point property, the conditional fixed point property for non-expansive mappings. See [8] for details. If C has normal structure, then it enjoys both properties.

Let int (C), the interior of a closed subset C of a Banach space E, be non-empty. A mapping $f: \mathrm{C} \to \mathrm{E}$ is said to satisfy the Leray-Schauder condition (actually due to Poincaré [17, p. 259]) if there is w in int (C) such that $f(y) - w \neq m(y - w)$ for all y in *bdy* (C) and $m > 1$. Here *bdy* (C) stands for the boundary of C.

PROBLEM 3. *Let* C, *a weakly compact convex subset of a Banach space* E, *have the fixed point property for non-expansive mappings. Let a non-expansive* $f: \mathrm{C} \to \mathrm{E}$ *satisfy the Leray-Schauder condition. Does f have a fixed point?*

The answer is known to be "yes" in the following cases:

(*a*) E is either uniformly convex [7, p. 661] or satisfies Opial's condition [31, p. 465];

(*b*) f satisfies a more restrictive boundary condition [29, p. 691];

(*c*) f satisfies an additional boundary condition [20];

(*d*) f is the restriction of a non-expansive self-mapping of a reflexive E which has normal structure [30, p. 263].

When (*d*) holds there is no need to assume that C is convex. Perhaps this is always true. At present, however, the answer to Problem 3 is not known even when C is assumed to have normal structure.

If x belongs to C, a convex subset of a Banach space E, we define $\mathrm{I_C}(x) = \{z \in \mathrm{E} : z = x + a(y - x) \text{ for some } y \in \mathrm{C} \text{ and } a \geq 0\}$. The closure of $\mathrm{I_C}(x)$ will be denoted by $cl(\mathrm{I_C}(x))$. If z belongs to E, then $z \in cl(\mathrm{I_C}(x))$ if and only if $\lim_{h \to 0+} \mathrm{d}((1 - h)x + hz, \mathrm{C})/h = 0$ [33]. Here $\mathrm{d}(w, \mathrm{C}) = \inf\{\|w - y\| : y \in \mathrm{C}\}$. A mapping $f: \mathrm{C} \to \mathrm{E}$ is said to be inward if $f(x) \in \mathrm{I_C}(x)$ for all $x \in \mathrm{C}$ and weakly inward if $f(x) \in \mathrm{cl}(\mathrm{I_C}(x))$ for all $x \in \mathrm{C}$ [16, p. 353].

PROBLEM 4. *Let* C *be a non-empty closed convex subset of a Banach space* E. *If a strict contraction* $f: \mathrm{C} \to \mathrm{E}$ *in weakly inward, then it has a unique fixed point* [24, p. 413]. *Is there an elementary proof of this fact?*

We ask this question because Martin's complicated proof is based on existence theorems for differential equations. The answer is known to be in the affirmative when E is isomorphic to a Hilbert space, or f is inward, or C is bounded and has a non-empty interior [33].

A related problem deals with k-set-contractions which were first considered by Darbo [10, p. 86]. These mappings satisfy $m(f(\mathrm{B})) \leq km(\mathrm{B})$ for some $k \geq 0$ and all bounded subsets B of their domain of definition. Here m denoted Kuratowski's measure of non-compactness [21, p. 303].

PROBLEM 5. *Let* C *be a bounded closed convex subset of a Banach space* E. *Let a weakly inward bounded continuous* $f: C \to E$ *be a k-contraction for some* $k < 1$. *Does* f *have a fixed point?*

The answer is known to be "yes" if f is inward [31, p. 462] or uniformly continuous [25, Theorem 11].

Let C, a convex boundedly weakly compact subset of a Banach space, possess normal structure. Let x_0 belong to C and let $T: C \to C$ be non-expansive. Consider the following two special Toeplitz iteration processes:

$$x_{n+1} = (1 - c_n) x_n + c_n Tx_n , \qquad n = 0, 1, 2, \cdots \tag{1}$$

$$x_{n+1} = (1 - k_n) x_0 + k_n Tx_n , \qquad n = 0, 1, 2, \cdots \tag{2}$$

In (1), $0 < c_n \leq 1$ for all n and $\sum_{i=0}^{\infty} c_i$ diverges. In (2), $0 < k_n \leq 1$ for all n and $\lim_{i \to \infty} k_i = 1$. It is known [32, p. 58], [30, p. 253] that if one of the sequences $\{x_n\}$ defined above is bounded, then T has a fixed point. (The converse statement is certainly true). However, (1) is not well-behaved with respect to convergence [18, p. 535]. On the other hand, the answer to our next problem is "yes" for Hilbert space [15] as well as for certain Banach spaces [32, p. 68].

PROBLEM. 6. *Let* E *be a Banach space. Is there a sequence* $\{k_n\}$ *such that whenever a weakly compact convex subset* C *of* E *possesses the fixed point property for non-expansive mappings, then the sequence* $\{x_n\}$ *defined by* (2) *converges to a fixed point of* T *for all* x_0 *in* C *and all non-expansive* $T: C \to C$*?*

Returning to the iterative process defined by (1) we recall [30, Theorem 1.1] which contains information on the asymptotic behavior of $\{x_n\}$. In a Banach space setting this theorem is not so satisfactory because it postulates a rather severe restriction on C. Therefore we pose the following question.

PROBLEM 7. *Can the restriction imposed on* C *in* [30, *Theorem 1.1*] *be weakened?*

An improvement of [30, Theorem 1.1] (with respect to the restrictions on the underlying Banach space) is obtained in [30, p. 253].

We turn now to set- valued mappings. If X is a topological space we denote by C (X) the set of all non-empty compact subsets of X. If X is a topological vector space we denote by CC (X) the set of all non-empty compact convex subsets of X. Finally, if X is a metric space we let CB (X) stand for the set of all non-empty closed and bounded subsets of X. The latter set becomes a metric space when it is equipped with the Hausdorff metric H. A mapping $F: (X, d) \to (CB(X), H)$ is called non-expansive if $H(F(x), F(y)) \leq \leq d(x, y)$ for all x and y in X.

PROBLEM 8. *Let* C, *a weakly compact convex subset of a Banach space* E, *possess the fixed point property for (single-valued) non-expansive mappings. Does every non-expansive* $F : C \to C(C)$ *have a fixed point?*

Again the answer is not known even when C is assumed to have normal structure (cfr. Problem 3). However, Lim [23] has recently shown that the answer is "yes" when E is uniformly convex in every direction. The same result has already appeared (with an incomplete proof) in [30, p. 253]. See [28, p. 29] for another partial result.

It is known [3, p. 459], [12, p. 607], [13] that if a (single-valued) self-mapping f of a complete metric space (X, d) satisfies $d(f(x), f(y)) \leq \leq k(d(x, y))\, d(x, y)$ for all x, y in $X, x \neq y$, where $k : (0, \infty) \to [0, 1)$ and (3) $\limsup_{r \to t+} k(r) < 1$ for all $t \in (0, \infty)$, then f has a (unique) fixed point. Therefore the following problem may be of interest.

PROBLEM 9. *Let* (X, d) *be a complete metric space. Suppose that* $F : X \to CB(X)$ *satisfies* $H(F(x), F(y)) \leq k(d(x, y))\, d(x, y)$ *for all* x, y *in* $X, x \neq y$, *where* $k : (0, \infty) \to [0, 1)$ *has property* (3). *Does* F *has a fixed point?*

F is known to possess a fixed point if $F : X \to C(X)$ [27, p. 40] or if k is a constant [26, p. 479]. See also [1, p. 30]. We remark in passing that [27, Proposition 1] is wrong [1, p. 33]. (This proposition was not used elsewhere).

If x belongs to C, a convex subset of a complex vector space E we denote the set $\{z \in E : z = x + c(y - x)$ for some $y \in C$ and c with $Re(c) > 1/2\}$ by $IF_C(x)$. Here $Re(c)$ is the real part of c. When E is real, then $IF_C(x) = I_C(x)$.

PROBLEM 10. *Let* C *be a non-empty compact convex subset of a locally convex Hausdorff topological vector space* E. *Let a mapping* $F : C \to CC(E)$ *be upper semi-continuous. If* $F(x) \subset IF_C(x)$ *for all* x *in* C, *does* F *have a fixed point?*

Again partial affirmative results are known. It has been established that F does have a fixed point when F is continuous [28, p. 21], [33]. (Actually, it is shown there that a slightly weaker continuity assumption is sufficient). This extends Fan's theorem for single-valued mappings [11, p. 235]. F certainly has a fixed point when E is real [6, p. 286].

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