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H. D. Pande, A. Kumar

Special infinitesimal projective transformation in a Finsler space

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Geometria differenziale. — *Special infinitesimal projective transformation in a Finsler space.* Nota [(*)] di H. D. PANDE e A. KUMAR, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Studio delle trasformazioni infinitesime proiettive speciali negli spazi di Finsler con l'uso della derivata di Lie.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n [1] [(1)] having fundamental metric tensor $g_{ij}(x, \dot{x})$ which is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \qquad \dot{\partial}_i \equiv \partial/\partial \dot{x}^i \tag{1.1a}$$

and

$$g_{ij} g^{jk} = \delta_i^k = \begin{cases} 1 \text{ if } k = i \\ 0 \text{ if } k \neq i . \end{cases} \tag{1.1b}$$

Let us consider, further, an infinitesimal point transformation

$$\overline{x}^i = x^i + u^i(x) \, dt \tag{1.2}$$

where $u^i(x)$ is a vector field and dt is an infinitesimal constant. The covariant derivative of a tensor field $T_j^i(x, \dot{x})$ is introduced in [1] as

$$T^i_{j|k} = \partial_k T^i_j - (\dot{\partial} m \, T^i_j) \, G^m_k + T^m_j \, \Gamma^{*i}_{mk} - T^i_m \, \Gamma^{*m}_{jk} \tag{1.3}$$

where $\Gamma^{*i}_{jk}(x, \dot{x})$ are the components of the connection parameter.

Using this covariant derivative and the infinitesimal point change (1.2), the Lie-derivatives of a tensor field and connection parameters are given by [2]:

$$\mathcal{L}_u T^i_j(x, \dot{x}) = T^i_{j|r} \, u^r + (\dot{\partial}_s T^i_j) \, u^s_{|r} \, \dot{x}^r - T^r_j \, u^i_{|r} + T^i_r \, u^r_{|j} \tag{1.4}$$

and

$$\mathcal{L}_u \Gamma^{*i}_{jk}(x, \dot{x}) = u^i_{|jk} + K^i_{jkr} \, u^r + (\dot{\partial}_r \Gamma^{*i}_{jk}) \, u^r_{|s} \, \dot{x}^s \tag{1.5}$$

respectively, where

$$K^i_{jkl}(x, \dot{x}) \overset{\text{def.}}{=} 2 \{ \partial_{[l} \Gamma^{*i}_{k]j} + (\dot{\partial}_r \Gamma^{*i}_{j[l}) \, G^r_{k]} + \Gamma^{*i}_{m[l} \Gamma^{*m}_{k]j} \}^2 \tag{1.6}$$

(*) Pervenuta all'Accademia il 14 ottobre 1974.

(1) The numbers in brackets refer to the references at the end of paper.

where $G^i(x, \dot{x})$ are the Berwald's connection parameters positively homogeneous of degree two in $\dot{x}^{i}$'s.

Between the operators $\mathcal{L}_u$, $\dot{\partial}$ and $|k$, we can obtain the following commutation formulae

$$(1.7)\qquad \dot{\partial}_l(\mathcal{L}_u T^i_j) - \mathcal{L}_u \dot{\partial}_l T^i_j = 0,$$

$$(1.8)\qquad (\mathcal{L}_u T^i_j)_{|r} - \mathcal{L}_u T^i_{j|r} = T^l_j \mathcal{L}_u \Gamma^{*i}_{rl} - T^i_l \mathcal{L}_u \Gamma^{*l}_{rj} - (\dot{\partial}_l T^i_j) \mathcal{L}_u \Gamma^{*l}_{rm} \dot{x}^m$$

and

$$(1.9)\qquad (\mathcal{L}_u \Gamma^{*i}_{jk})_{|l} - (\mathcal{L}_u \Gamma^{*i}_{jl})_{|k} = \mathcal{L}_u K^i_{jkl} + 2(\dot{\partial}_r \Gamma^{*i}_{j[k}) \mathcal{L}_u \Gamma^{*r}_{l]s} \dot{x}^s.$$

The projective deviation tensor field $W^i_j(x, \dot{x})$ is given by:

$$(1.10)\qquad W^i_j(x, \dot{x}) = H^i_j - H\delta^i_j - \dot{x}^i(\dot{\partial}_r H_j - \dot{\partial}H)/(n+1),$$

and satisfies the following identities:

$$(1.11)\qquad a)\ W^i_l \dot{x}^l = 0, \quad b)\ \dot{\partial}_l W^i_j \dot{x}^l = 2W^i_j, \quad c)\ W^i_i = 0, \quad d)\ \dot{\partial}_i W^i_j = 0.$$

2. SPECIAL INFINITESIMAL PROJECTIVE TRANSFORMATION

Under the projective transformation defined by

$$(2.1)\qquad \bar{G}^i(x, \dot{x}) = G^i(x, \dot{x}) - P(x, \dot{x})\dot{x}^i$$

where $P(x, \dot{x})$ is a scalar function, Sinha [3] has considered the Lie-derivative of a connection parameter $\Gamma^{*i}_{jk}(x, \dot{x})$ as

$$(2.2)\qquad \mathcal{L}_u \Gamma^{*i}_{jk}(x, \dot{x}) = -2\delta^i_{(j}\dot{\partial}_{k)} P.$$

Under the special projective transformation characterized by

$$(2.3)\qquad P = \frac{1}{(n+1)} G^r_{rs}\dot{x}^s \quad , \quad \dot{\partial}_k P = \frac{1}{(n+1)} G^r_{rk}$$

we have

$$(2.4)\qquad \mathcal{L}_u \Gamma^{*i}_{jk}(x, \dot{x}) = -\frac{2}{(n+1)} \delta^i_{(k} G^r_{k)r}.$$

Substituting the value of $\mathcal{L}_u \Gamma^{*i}_{jk}(x, \dot{x})$ in the commutation formula (1.9), we get

$$(2.5)\qquad \mathcal{L}_u K^i_{jkl}(x, \dot{x}) = -\frac{2}{(n+1)}[\delta^i_j G^r_{r[k|l]} + G^r_{rj|[l}\delta^i_{k]} - (\dot{\partial}_a \Gamma^{*i}_{j[k})(\delta^a_{l]} G^r_r + G^r_{l]r}\dot{x}^a)].$$

(2) $2A_{[hk]} = A_{hk} - A_{kh}$, $2A_{(hk)} = A_{hk} + A_{kh}$.

Multiplying (2.5) by $\dot{x}^j \dot{x}^k$ and noting the homogeneity property of $G^i(x, \dot{x})$, we obtain

$$(2.6) \qquad \mathcal{L}_u K^i_{jkl} \dot{x}^j \dot{x}^k = -\frac{1}{(n+1)} [2 G^r_{r|l} \dot{x}^i - G^r_{rl|k} \dot{x}^i \dot{x}^k -$$

$$- \delta^i_l G^r_{r|k} \dot{x}^k - \{(\dot{\partial}_l \Gamma^{*i}_{jk}) G^r_r + (\dot{\partial}_a \Gamma^{*i}_{jk}) G^r_{rl} \dot{x}^a\} \dot{x}^j \dot{x}^k - 2 (\dot{\partial}_a \Gamma^{*i}_{jl}) G^r_r \dot{x}^a \dot{x}^j]$$

where we have the fact that $\mathcal{L}_u \dot{x}^i = \dot{x}^i{}_{|k} = 0$.

Contracting (2.5) with respect to the indices i and l, we get

$$(2.7) \qquad \mathcal{L}_u K_{jk} = -\frac{1}{(n+1)} [G^r_{rk|j} - n G^r_{rj|k} - (\dot{\partial}_i \Gamma^{*i}_{jk}) G^r_r -$$

$$- (\dot{\partial}_a \Gamma^{*i}_{jk}) G^r_{ri} \dot{x}^a + (\dot{\partial}_k \Gamma^{*i}_{ji}) G^r_r + (\dot{\partial}_a \Gamma^{*i}_{ji}) G^r_{rk} \dot{x}^a].$$

Multiplying (2.7) by $\dot{x}^j \dot{x}^k$, we have

$$(2.8) \qquad \mathcal{L}_u K_{jk} \dot{x}^j \dot{x}^k = -\frac{1}{(n+1)} [(1-n) G^r_{r|k} \dot{x}^k - \dot{x}^j \{(\dot{\partial}_i \Gamma^{*i}_{jk}) G^r_r \dot{x}^k +$$

$$+ (\dot{\partial}_a \Gamma^{*i}_{jk}) G^r_{ri} \dot{x}^a - (\dot{\partial}_k \Gamma^{*i}_{ji}) G^r_r \dot{x}^k - (\dot{\partial}_a \Gamma^{*i}_{ji}) G^r_r \dot{x}^a\}).$$

Eliminating the term $G^r_{r|k} \dot{x}^k$ from the equations (2.6) and (2.8) we obtain

$$(2.9) \qquad \mathcal{L}_u M^i_l (x, \dot{x}) = \frac{1}{(n+1)} [(n-1)\{2 G^r_{r|l} \dot{x}^i - G^r_{rl|k} \dot{x}^i \dot{x}^k -$$

$$- n \{(\dot{\partial}_l \Gamma^{*i}_{jk}) G^r_r + (\dot{\partial}_a \Gamma^{*i}_{jk}) G^r_{rl} \dot{x}^a + 2 (\dot{\partial}_k \Gamma^{*i}_{jl}) G^r_r\} \dot{x}^k \dot{x}^j]$$

where

$$(2.10) \qquad M^i_l (x, \dot{x}) \overset{\text{def.}}{=} (1-n) \mathcal{L}_u K^i_{jkl} \dot{x}^j \dot{x}^k - \delta^i_l \mathcal{L}_u K_{jk} \dot{x}^j \dot{x}^k .$$

3. Some theorems

Applying the commutation formula (1.8) to the projective deviation tensor field $W^i_j (x, \dot{x})$, we get

$$(3.1) \qquad \mathcal{L}_u W^i_{j|r} - (\mathcal{L}_u W^i_j)_{|r} = -\frac{1}{(n+1)} [2\{W^l_j \delta^i_{(r} G^m_{l)m} - W^i_l \delta^l_{(r} G^m_{j)m}\} -$$

$$- (\dot{\partial}_r W^i_j) G^m_m - 2 W^i_j G^m_{mr}] .$$

Contracting (3.1) with respect to the indices i and r and in view of (1.11), we get

$$(3.2) \qquad \mathcal{L}_u W^i_{j|i} - (\mathcal{L}_u W^i_j)_{|i} = - W^l_j G^m_{ml} .$$

Transvecting (3.1) by $\dot{x}^r$ and noting (1.11), we have

$$(3.3) \qquad \{\mathcal{L}_u W^i_{j|r} - (\mathcal{L}_u W^i_j)_{|r}\} \dot{x}^r = -\frac{1}{(n+1)} \{W^l_j G^m_{ml} \dot{x}^i - 4 W^i_j G^m_m\} .$$

We now suppose that the special infinitesimal projective transformation leaves invariant the covariant derivative of projective deviation tensor, i.e.

$$\mathcal{L}_u W^i_{j|r} = 0 . \tag{3.4}$$

In view of (3.4), equations (3.2) and (3.3) reduce to

$$(\mathcal{L}_u W^i_j)_{|i} - W^l_j G^m_{ml} = 0 \tag{3.5}$$

and

$$(n+1)(\mathcal{L}_u W^i_j)_{|r} \dot{x}^r = W^l_j G^m_{ml} \dot{x}^i - 4 W^i_j G^m_m . \tag{3.6}$$

Eliminating $W^l_j G^m_{ml}$ from equations (3.5) and 3.6), we have

$$(n+1)(\mathcal{L}_u W^i_j)_{|r} \dot{x}^r - (\mathcal{L}_u W^r_j)_{|r} \dot{x}^i + 4 W^i_j G^m_m = 0. \tag{3.7}$$

Thus, we have

THEOREM 2.1. *If a Finsler space* F_n *admits an non-affine special infinitesimal projective transformation such that the covariant derivative of* $W^i_j(x, \dot{x})$ *remains invariant then equation (3.7) holds.*

THEOREM 3.2. *If the special infinitesimal projective transformation is affine in a Finsler space such that the covariant derivative of* W^i_j *remains invariant then we have*

$$(n+1)(\mathcal{L}_u W^i_j)_{|r} \dot{x}^r = (\mathcal{L}_u W^r_j)_{|r} \dot{x}^i . \tag{3.8}$$

In a symmetric Finsler space [5] i.e. $W^i_{k|r} = 0$, equation (3.4) is always satisfied; then we have

THEOREM 3.3. *If a symmetric Finsler space admits an non-affine special infinitesimal projective transformation then (3.7) necessarily holds.*

THEOREM 3.4. *If the special infinitesimal projective transformation is affine in a symmetric Finsler space then (3.8) necessarily holds.*

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