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Masahiro Kon

Totally real minimal submanifolds with parallel second fundamental form

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Geometria differenziale. — *Totally real minimal submanifolds with parallel second fundamental form.* Nota (*) di MASAHIRO KON, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Caratterizzazione di una sottovarietà minima totalmente reale immersa in uno spazio proiettivo complesso con curvatura sezionale olomorfa costante.

1. INTRODUCTION

Let $\bar{M}^n(c)$ be a Kaehler manifold of real dimension 2n and constant holomorphic sectional curvature c. A submanifold M of $\bar{M}^n(c)$ is *totally real* if $T_m(M)$ is perpendicular to $JT_m(M)$ for every $m \in M$, where J denotes the complex structure and $T_m(M)$ is the tangent space of M at m. We denote by $P^n(c)$ an n-dimensional complex projective space of constant holomorphic sectional curvature c. The purpose of this note is to prove the following

THEOREM. *Let* M *be an n-dimensional totally real minimal submanifold immersed* in $P^n(c)$ *with parallel second fundamental form. Then either* M *is totally geodesic or* M *has the nonnegative scalar curvature* $K \geqq 0$. *Moreover if* $K = 0$, *then* M *is flat.*

2. PRELIMINARIES

Let M be an n-dimensional totally real minimal submanifold of $\bar{M}^n(c)$. We denote by $\langle\, ,\rangle$ the metric tensor field of $\bar{M}^n(c)$, as well as the metric induced on M. Let $\bar{\nabla}$ and ∇ be the covariant differentiation of $\bar{M}$ and M respectively. Then the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) \quad , \quad \bar{\nabla}_X N = -A^N(X) + D_X N$$

for any tangent vector fields X and Y on M and normal vector field N on M, where D is the linear connection in the normal bundle $T(M)^\perp$. Both A and B are called the second fundamental form of M and they satisfy $\langle B(X, Y), N\rangle = \langle A^N(X), Y\rangle$. For B we define its covariant derivative $\tilde{\nabla}_X B$ by setting

$$(\tilde{\nabla}_X B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

If $\tilde{\nabla} B = 0$, the second fundamental form is said to be *parallel*. Let $e_1, \cdots, e_n$ be an orthonormal frame for $T_m(M)$. If $\sum_{i=1}^{n} B(e_i, e_i) = 0$, then M is said to be *minimal* and if $B = 0$, then M is said to be totally *geodesic*. We

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denote by R the Riemannian curvature tensor of M. Since M is totally real, we obtain

$$R_{X,Y} Z = \frac{1}{4} c(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + A^{B(Y,Z)}(X) - A^{B(X,Z)}(Y). \tag{2.1}$$

Now we can choose a normal frame $Je_1, \cdots, Je_n$ for $T_m(M)^{\perp}$. Then the Ricci operator Q of M is given by

$$Q = \frac{1}{4}(n-1) cI - \sum_{i=1}^{n} (A^{i*})^2, \tag{2.2}$$

where I is the identity tensor and $A^{i*} \equiv A^{Je_i}$. And the scalar curvature K of M is given by

$$K = \frac{1}{4} n(n-1) c - \| A \|^2, \tag{2.3}$$

where $\| A \|$ denotes the length of the second fundamental form. In our case we have the relation [1]:

$$A^{JX}(Y) = A^{JY}(X). \tag{2.4}$$

In this note we use the following two symmetric, positive semi-definite operators defined by Simons [5]:

$$\tilde{A} = {}^t A \circ A \quad , \quad \underset{\sim}{A} = \sum_{i=1}^{n} a dA^{i*} a dA^{i*}.$$

3. Proof of theorem

First we prepare some formulas for an n–dimensional totally real minimal submanifold M of $\bar{M}^n(c)$. We define that $\| Q \|^2 = \sum_{i=1}^{n} \langle Qe_i, Qe_i \rangle$ and $\| R \|^2 = \sum_{i,j,k=1}^{n} \langle R_{e_i,e_j} e_k, R_{e_i,e_j} e_k \rangle$. It is well known that $\| Q \|^2 \geqq K^2/n$ and $\| R \|^2 \geqq 2 K^2/n(n-1)$ and the first equality holds if and only if M is Einstein and the second equality holds if and only if M is of constant curvature.

From (2.2), (2.3) and (2.4) we obtain

$$\begin{aligned} \langle A \circ \tilde{A}, A \rangle &= \sum_{i,j=1}^{n} (\mathrm{Tr} A^{i*} A^{j*})^2 = \mathrm{Tr}\left(\sum_{i=1}^{n} (A^{i*})^2\right)^2 \\ &= \frac{1}{n} \| A \|^4 + \| Q \|^2 - \frac{1}{n} K^2. \end{aligned} \tag{3.1}$$

And we have, by using (2.1), (2.3) and (2.4),

$$\begin{aligned} \langle \underset{\sim}{A} \circ A, A \rangle &= \sum_{i,j=1}^{n} \| [A^{i*}, A^{j*}] \|^2 \\ &= \frac{2}{n(n-1)} \| A \|^4 + \| R \|^2 - \frac{2}{n(n-1)} K^2. \end{aligned} \tag{3.2}$$

By the definition, $\langle \underset{\sim}{A} \circ A , A \rangle$ is just the *scalar normal curvature* K_N of M (cfr. [2]) and we have the following

PROPOSITION. *Let* M *be an n-dimensional totally real minimal submanifold of* $\bar{M}^n(c)$. *Then* $K_N \geqq 2 \| A \|^4 / n(n-1)$ *and equality holds if and only if* M *is of constant curvature.*

Now we shall prove our theorem. Since the second fundamental form of M is parallel, the Simons' type formula for A is given by (cfr. [1, 3]):

$$\langle A \circ \tilde{A} , A \rangle + \langle \underset{\sim}{A} \circ A , A \rangle - \frac{1}{4}(n+1)\, c \, \| A \|^2 = 0. \tag{3.3}$$

Therefore (2.3), (3.1), (3.2) and (3.3) imply

$$\frac{(n+1)}{n(n-1)} \| A \|^2 K = \| R \|^2 - \frac{2}{n(n-1)} K^2 + \| Q \|^2 - \frac{1}{n} K^2 \geqq 0 , \tag{3.4}$$

from which we deduces that either $\| A \|^2 = 0$ (i.e., M is totally geodesic) or $K \geqq 0$. Moreover if $K = 0$, then M is of constant curvature by (3.4) and hence M is flat. This completes our theorem.

Remark 1. There exists an example of a flat totally real minimal submanifold with parallel second fundamental form which is not totally geodesic. Let S^1 be a unit sphere of dimension 1. Then $S^1 \times S^1$ is a compact, minimal, totally real surface with parallel second fundamental form immersed in $P^2(4)$ (see [3, 4]).

Remark 2. Let M be an *n*-dimensional totally real minimal submanifold of $\bar{M}^n(c)$ with parallel second fundamental form. If $c \leqq 0$, then M is obviously totally geodesic, by using (3.1), (3.2) and (3.3). Consequently our theorem has the meaning when the ambient space $\bar{M}^n(c)$ is *elliptic*, i.e., $c > 0$. Hence we have proved our theorem for $P^n(c)$.

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