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Projective transformation in recurrent and Ricci-recurrent Finsler spaces

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Geometria differenziale. — *Projective transformation in recurrent and Ricci-recurrent Finsler spaces.* Nota (*) di S. C. SRIVASTAVA e R. S. SINHA, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — La nozione di trasformazione proiettiva già nota per gli spazi riemanniani ricorrenti rispetto al tensore di curvatura o a quello di Ricci viene estesa agli spazi di Finsler con analoghe proprietà.

1. INTRODUCTION

Pravanovitch [4] (1) studied the projective and conformal transformations in a recurrent and Ricci-recurrent Riemannian space and in recent papers Sinha [5], [6], studied projective transformations for Finsler spaces. In this paper the authors wish to study this transformation in the Finsler space F_n with recurrent and Ricci-recurrent curvature.

Consider a Finsler space F_n of n-dimensions with the connection coefficient G^i_{jk}. If its curvature tensor H^i_{jhk} satisfies the relation

$$H^i_{jhk(l)} = K_l H^i_{jhk}, \qquad (i, j, k, \cdots = 1, 2, \cdots, n) \tag{1.1}$$

where suffix in bracket denotes covariant differentiation with respect to G^i_{jk} and $K_l = K_{(l)}$ is a non–zero vector, then F_n is called a recurrent Finsler space [1]. It is to be noted that K_l is homogeneous of degree zero in $\dot{x}^i$ (2). In this we denote such a Finsler space by F^*_n. However if the Ricci tensor $H_{ij} = H^h_{ijh}$ satisfies

$$H_{ij(l)} = K_l H_{ij} \tag{1.2}$$

then F^*_n is a Ricci-recurrent Finsler space. If $K_l = 0$, i.e. $H_{ij(l)} = 0$ and $H^i_{jhk(l)} = 0$, F^*_n is a Ricci-symmetric or symmetric Finsler space respectively. The curvature tensor H^h_{ijk} satisfies the following relations [2]

$$H^h_{ijk}\dot{x}^i = H^h_{jk} \quad ; \quad H^h_{ijh} = H_{ij} \tag{1.3}$$

$$H_{ij}\dot{x}^i = H_j \tag{1.4}$$

$$H_{ij}\dot{x}^j = (n-1)\dot{\partial}_i H - H_i \tag{1.5}$$

$$H_{ij}\dot{x}^i\dot{x}^j = (n-1) H \tag{1.6}$$

where $H_i = H^h_{ih}$ and $H = \frac{1}{n-1} H^i_i$.

(*) Pervenuta all'Accademia il 14 ottobre 1974.

(1) Numbers in the square brackets refer to the references at the end of the paper.

(2) $\dot{x}^i = \dfrac{dx^i}{dt}$; $\dot{\partial}_i = \partial/\partial\dot{x}^i$, $2\,T_{[ij]} = T_{ij} - T_{ji}$ and $2\,T_{(ij)} = T_{ij} + T_{ji}$.

It can be noted that H^h_{ijk} is homogeneous of degree zero, consequently H_{ij} is also homogeneous of degree zero. So we have

$$\dot{\partial}_r H_{ij} \dot{x}^r = 0. \tag{1.7}$$

In F_n, let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, d\tau \tag{1.8}$$

where v^i denote components of a contravariant vector field defined over the domain of the space under consideration and $d\tau$ an infinitesimal constant. If a Finsler space F_n admits infinitesimal projective transformation with respect to the vector field v^i, then the Lie derivative of G^i_{jh} with respect to (1.8) has the form [5]

$$\mathcal{L} G^i_{jh} = -2\, \delta^i_{(j}\, \dot{\partial}_{h)} P - (\dot{\partial}_j \dot{\partial}_h P)\, \dot{x}^i \tag{1.9}$$

where $P(x, \dot{x})$ is an arbitrary scalar function positively homogeneous of degree one in $\dot{x}^i$. Due to this homogeneity property

$$\dot{\partial}_r P \dot{x}^r = P \qquad \text{and} \qquad \dot{\partial}_r \dot{\partial}_l P \dot{x}^r = 0. \tag{1.10}$$

The Lie derivative of the curvature tensor H^i_{jhk} is given by

$$\mathcal{L} H^i_{jhk} = 2\, \{\delta^i_h\, \dot{\partial}_{[k}\, P_{(j)]} + \dot{\partial}_h P_{[(j)}\, \delta^i_{k]} + (\dot{\partial}_h\, \dot{\partial}_{[k}\, P)_{(j)]}\, \dot{x}^i\}. \tag{1.11}$$

From equation (1.11) we have [6].

$$\mathcal{L} H^i_{hjk}\, \dot{x}^h = \mathcal{L} H^i_{jk} \tag{1.12}$$

$$\mathcal{L} H_{hj} = n \dot{\partial}_h P_{(j)} - \dot{\partial}_j P_{(h)} - (\dot{\partial}_h \dot{\partial}_j P)_{(i)}\, \dot{x}^i = \Phi_{hj} \quad \text{(say)}. \tag{1.13}$$

and

$$\mathcal{L} H = P_{(i)}\, \dot{x}^i. \tag{1.14}$$

Sinha proved [5]

THEOREM. *If an F^*_n space $(n \geq 0)$ admits an infinitesimal projective motion, the motion should be of the form*

$$\mathcal{L} G^i_{jh} = -2\, \delta^i_{(j}\, \dot{\partial}_{h)} P - (\dot{\partial}_j \dot{\partial}_h P)\, \dot{x}^i \quad , \quad 4P = \mathcal{L} K_l\, \dot{x}^l. \tag{1.15}$$

For a Ricci-symmetric and symmetric Finsler space $K_l = 0$, so from the above theorem $P = 0$ which shows that the transformation is affine. So we have the theorem.

THEOREM 1.1. *If a symmetric Finsler space admits an infinitesimal projective transformation, then it is also affine.*

2. PROJECTIVE TRANSFORMATION IN RECURRENT AND RICCI-RECURRENT FINSLER SPACE

Taking covariant derivative of (1.2), we have

$$H_{ij(l)(k)} = K_{l(k)} H_{ij} + K_l K_k H_{ij} .$$

On interchanging l and k in the above relation and subtracting the equation thus obtained, we have

$$H_{ij(l)(k)} - H_{ij(k)(l)} = \{K_{l(k)} - K_{k(l)}\} H_{ij} .$$

By taking into consideration

$$K_{l(k)} - K_{k(l)} = - \dot{\partial}_r K H^r_{lk}$$

the above relation becomes

$$H_{ij(l)(k)} - H_{ij(k)(l)} = - \dot{\partial}_r K H^r_{lk} H_{ij} .$$

On applying the Ricci identity, the above relation becomes

$$\dot{\partial}_r H_{ij} H^r_{lk} + H_{rj} H^r_{ilk} + H_{ir} H^r_{jlk} = \dot{\partial}_r K H^r_{lk} H_{ij} . \tag{2.1}$$

Contracting (2.1) with $\dot{x}^i \dot{x}^j$ and using (1.3), (1.4), (1.5) and (1.6) we have

$$H^r_{lk} \{\dot{\partial}_r H - H \dot{\partial}_r K\} = 0 \tag{2.2}$$

that is,

$$(a) \quad H^r_{lk} = 0 \qquad \text{or} \quad (b) \quad \dot{\partial}_r H - H \dot{\partial}_r K = 0. \tag{2.3}$$

Hence we have the theorem

THEOREM 2.1. *The Ricci-recurrent Finsler space, admitting infinitesimal projective transformation satisfies the relation* $H^r_{lk} = 0$ *or* $\dot{\partial}_r H - H \dot{\partial}_r K = 0$.

Taking the Lie derivative of (2.1) and using (1.7), (1.10), (1.11), (1.12) and (1.13) we have

$$\begin{aligned} & \dot{\partial}_r \Phi_{ij} H^r_{lk} + \dot{\partial}_k H_{ij} P_{(l)} - \dot{\partial}_l H_{ij} P_{(k)} + \Phi_{rj} H^r_{ilk} + \\ & + 2 H_{ij} \{\dot{\partial}_k P_{(l)} - \dot{\partial}_l P_{(k)}\} + H_{kj} \dot{\partial}_i P_{(l)} - H_{lj} \dot{\partial}_i P_{(k)} + \\ & + H_j \{(\dot{\partial}_i \dot{\partial}_k P)_{(l)} - (\dot{\partial}_i \dot{\partial}_l P)_{(k)}\} + \Phi_{ir} H^r_{jlk} + H_{ik} \dot{\partial}_j P_{(l)} - \\ & - H_{il} \dot{\partial}_j P_{(k)} + \{(n-1) \dot{\partial}_i H - H_i\} \{(\dot{\partial}_j \dot{\partial}_k P)_{(l)} - (\dot{\partial}_j \dot{\partial}_l P)_{(k)}\} = \\ & = (\mathscr{L} \dot{\partial}_r K) H^r_{lk} H_{ij} + \{\dot{\partial}_k K P_{(l)} - \dot{\partial}_l K P_{(k)}\} H_{ij} + \dot{\partial}_r K H^r_{lk} \Phi_{ij} . \end{aligned} \tag{2.4}$$

Contracting (2.4) with $\dot{x}^i \dot{x}^j$ and using (1.3), (1.6), (1.7) and (1.10) we have

$$
\begin{aligned}
(2.5) \qquad & \dot{\partial}_r \Phi_{ij} H^r_{lk} \dot{x}^i \dot{x}^j + \dot{\partial}_k H_{ij} P_{(l)} \dot{x}^i \dot{x}^j - \dot{\partial}_l H_{ij} P_{(k)} \dot{x}^i \dot{x}^j + \\
& + \Phi_{rj} H^r_{lk} \dot{x}^j + 2(n-1) H \{\dot{\partial}_k P_{(l)} - \dot{\partial}_l P_{(k)}\} + H_{kj} P_{(l)} \dot{x}^j - \\
& - H_{lj} P_{(k)} \dot{x}^j + \Phi_{ir} H^r_{lk} \dot{x}^i + H_{ik} P_{(l)} \dot{x}^i - H_{il} P_{(k)} \dot{x}^i \\
& = (\mathcal{L} \dot{\partial}_r K) H^r_{lk} (n-1) H + (n-1) H \{\dot{\partial}_k KP_{(l)} - \dot{\partial}_l KP_{(k)}\} + \\
& + \dot{\partial}_r KH^r_{lk} \Phi_{ij} \dot{x}^i \dot{x}^j .
\end{aligned}
$$

From (1.4), (1.5), (1.6) and (1.13) we can easily deduce the following relations

$$(2.6) \qquad \dot{\partial}_r H_{ij} \dot{x}^i \dot{x}^j = 0$$

$$(2.7) \qquad \Phi_{ij} \dot{x}^i = n P_{(j)} - \dot{\partial}_j P_{(i)} \dot{x}^i$$

$$(2.8) \qquad \Phi_{ij} \dot{x}^j = n \dot{\partial}_i P_{(j)} \dot{x}^j - P_{(i)}$$

$$(2.9) \qquad \Phi_{ij} \dot{x}^i \dot{x}^j = (n-1) P_{(i)} \dot{x}^i$$

and

$$(2.10) \qquad \dot{\partial}_r \Phi_{ij} \dot{x}^i \dot{x}^j = 0 .$$

Using (1.4), (2.6), (2.7), (2.8), (2.9) and (2.10), the equation (2.5) reduces to

$$
\begin{aligned}
(2.11) \qquad & \{\dot{\partial}_r P_{(j)} \dot{x}^j + P_{(r)}\} H^r_{lk} + 2 H \{\dot{\partial}_k P_{(l)} - \dot{\partial}_l P_{(k)}\} + \dot{\partial}_k HP_{(l)} - \dot{\partial}_l HP_{(k)} = \\
& = (\mathcal{L} \dot{\partial}_r K) HH^r_{lk} + H \{\dot{\partial}_k KP_{(l)} - \dot{\partial}_l KP_{(k)}\} + \dot{\partial}_r KH^r_{lk} P_{(i)} \dot{x}^i .
\end{aligned}
$$

Now taking the Lie derivative of (2.3), we have from (1.14)

$$(2.12) \qquad H \mathcal{L} \dot{\partial}_r K = \dot{\partial}_r P_{(j)} \dot{x}^j + P_{(r)} - \dot{\partial}_r KP_{(j)} \dot{x}^j .$$

On account of (2.12) the equation (2.11) becomes

$$H \dot{\partial}_{[k} P_{(l)]} = 0 , \qquad \text{i.e.} \quad H = 0 \qquad \text{or} \quad \dot{\partial}_{[k} P_{(l)]} = 0 .$$

So we have the following

THEOREM 2.2. *The Ricci-recurrent Finsler space, admitting infinitesimal projective transformation satisfies either* $H = 0$ *or the scalar* $P(x, \dot{x})$ *satisfies the relation* $\dot{\partial}_{[k} P_{(l)]} = 0$.

Since every recurrent Finsler space is Ricci-recurrent, we have following special cases of Theorems 2.1 and 2.2.

THEOREM 2.3. *The recurrent Finsler space* F^*_n *admitting infinitesimal projective transformation satisfies the relation* $H^r_{lk} = 0$ *or* $\dot{\partial}_r H - H \dot{\partial}_r K = 0$.

THEOREM 2.4. *The recurrent Finsler space* F_n^* *admitting infinitesimal projective transformation satisfies the relation* $H = 0$ *or* $\dot{\partial}_{[k} P_{(l)]} = 0$.

Combining the Theorems 2.3 and 2.4 we conclude

THEOREM 2.5. *If the recurrent Finsler space* F_n^* *admits infinitesimal projective transformation, one of the following conditions must be satisfied.*

(1) $H^r_{lk} = 0$;

(2) *the scalar* $H = 0$;

(3) *the scalar* K *satisfies* $\dot{\partial}_r H - H \dot{\partial}_r K = 0$;

(4) *The scalar* P *satisfies* $\dot{\partial}_{[k} P_{(l)]} = 0$.

3. SOME THEOREMS

Taking Lie derivative of (1.2) we have

$$\mathcal{L} H_{ij(l)} = (\mathcal{L} K_l) H_{ij} + K_l \mathcal{L} H_{ij}. \tag{3.1}$$

It is well known that [3]

$$\mathcal{L} T^i_{j(r)} - (\mathcal{L} T^i_j)_{(r)} = (\mathcal{L} G^i_{rl}) T^l_j - (\mathcal{L} G^l_{rj}) T^i_l - (\mathcal{L} G^l_{rm}) \dot{x}^m \dot{\partial}_l T^i_j \tag{3.2}$$

where $T^i_j(x, \dot{x})$ is any tensor.

With the help of (3.2) we have

$$\mathcal{L} H_{ij(l)} = (\mathcal{L} H_{ij})_{(l)} - H_{rj} \mathcal{L} G^r_{il} - H_{ir} \mathcal{L} G^r_{jl} - \dot{\partial}_r H_{ij} \dot{x}^m \mathcal{L} G^r_{lm}$$

which with the help of (1.9), (1.13) and (3.1) becomes

$$H_{ij} \mathcal{L} K_l + K_l \Phi_{ij} = \Phi_{ij(l)} - H_{lj} \dot{\partial}_i P - 2 \dot{\partial}_l P H_{ij} - H_{il} \dot{\partial}_j P -$$

$$- H_{rj} \dot{\partial}_l \dot{\partial}_i P \dot{x}^r - H_{ir} \dot{\partial}_l \dot{\partial}_j P \dot{x}^r - P \dot{\partial}_l H_{ij}.$$

Contracting with $\dot{x}^i \dot{x}^j$ and using (1.3), (1.4), (1.5), (1.11) and (2.9) we have

$$\{P_{(i)(l)} - K_l P_{(i)}\} \dot{x}^i - P \dot{\partial}_l H - H (2 \dot{\partial}_l P + \mathcal{L} K_l) = 0. \tag{3.3}$$

Contracting (3.3) with $\dot{x}^l$ and using (1.15) we have

$$\{P_{(i)(l)} - K_l P_{(i)}\} \dot{x}^i \dot{x}^l - 8 PH = 0. \tag{3.4}$$

Hence we have the following theorems:

THEOREM 3.1. *If a Ricci-recurrent Finsler space admits infinitesimal projective transformation then*

$$\{P_{(i)(l)} - K_l P_{(i)}\} \dot{x}^i \dot{x}^l - 8 PH = 0.$$

THEOREM 3.2. *If a Ricci-recurrent Finsler space admitting a projective transformation satisfies* H = 0 *then*

$$P_{(i)(l)} - K_l\, P_{(i)} = 0.$$

The above theorems are also true for recurrent Finsler space F_n^*.

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