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Remarks on a totally real submanifold in almost Tachibana manifolds

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Geometria differenziale. — *Remarks on a totally real submanifold in almost Tachibana manifolds.* Nota (*) di SEIICHI YAMAGUCHI, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Le sottovarietà di una varietà quasi hermitiana si distinguono in due classi *olomorfe* e *antiolomorfe* (o *totalmente reali*). Definiti le varietà « quasi di Tachibana » se ne studiano le sottovarietà totalmente reali.

§ 1. INTRODUCTION

C. S. Houh [3], S. T. Yau [8], B. Y. Ohen and K. Cgiue [1] have investigated totally real (anti-holomorphic) submanifolds in an almost Hermitian manifold or a Kahlerian manifold of constant holomorphic sectional curvature and obtained many interesting results.

Recently, the author and T. Ikawa [6] have proved the following:

THEOREM. *Let* M^n *be a totally real submanifold of a Kählerian manifold* $\bar{M}^{2n}$. *A necessary and sufficient condition in order that the normal connection is flat is that the submanifold* M^n *is flat.*

The purpose of this paper is to study a totally real submanifold in almost Tachibana manifolds, that is, we prove the following:

THEOREM 1. *Let* M^n *be a totally real submanifold of an almost Tachibana manifold* $\bar{M}^{2n}$. *A necessary and sufficient condition in order that the normal connection is flat is that*

$$\bar{R}(Z, Y) JX - J\bar{R}(Z, Y) X + JR(Z, Y) X = 0 .$$

THEOREM 2. *Let* M^3 *be a totally real submanifold of a special almost Tachibana manifold* $\bar{M}^6$ *with constant* α. *A necessary and sufficient condition in order that the normal connection is flat that the submanifold* M^3 *is of constant curvature* a.

§ 2. ALMOST TACHIBANA MANIFOLD

Let M be a C^∞ almost Hermitian manifold with metric tensor $\langle , \rangle$, Riemannian connection ∇, and almost complex structure J. Denote by $\mathfrak{X}(M)$ the vector fields of M. Then M is said to be an almost Tachibana manifold (K–space, nearly Kähler manifold) provided $(\nabla_X J)(X) = 0$ for all $X \in \mathfrak{X}(M)$.

(*) Pervenuta all'Accademia il 6 settembre 1974.

The curvature tensor $R(X, Y)$ $(X, Y \in \mathfrak{X}(M))$ of an almost Tachibana manifold satisfies [2,5]

$$(2.1) \qquad \langle R(Z, Y) X, W\rangle - \langle R(Z, Y) JX, JW\rangle = -\langle (\nabla_Z J)(Y), (\nabla_X J)(W)\rangle, \quad W, X, Y, Z \in \mathfrak{X}(M).$$

We put $\Omega(X, Y) = \langle JX, Y\rangle$ for all $X, Y \in \mathfrak{X}(M)$. Then the 2-form Ω is said to be associated. An almost Tachibana manifold M is said to be special with constant α if the associated form Ω is a special Killing 2-form with constant α $(\neq 0)$ [5].

In a special almost Tachibana manifold with constant α, we have [5]

$$(2.2) \qquad \langle (\nabla_Z J)(Y), (\nabla_X J)(W)\rangle = -\alpha(\langle Z, X\rangle\langle Y, W\rangle - \langle Z, W\rangle\langle Y, X\rangle - \langle JZ, X\rangle\langle JY, W\rangle + \langle JZ, W\rangle\langle JY, X\rangle).$$

It is known that a special almost Tachibana manifold is a 6-dimensional Einstein manifold [5].

§ 3. Submanifold

Let M^n be a submanifold immersed in a Riemannian manifold M^{n+p}. Let $\langle\, ,\rangle$ be the metric tensor on $\bar{M}^{n+p}$ as well as the metric tensor induced on M^n. We denote by $\bar{\nabla}$ the Riemannian connection in $\bar{M}^{n+p}$ and ∇ the Riemannian connection in M^n determined by the induced metric on M^n. Let $\mathfrak{X}^{\perp}(M^n)$ be the set of all vector fields normal to M^n.

The Gauss-Weingarten formulas are given by

$$(3.1) \qquad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)$$

$$(3.2) \qquad \bar{\nabla}_X N = -A^N(X) + D_X N \quad , \quad X, Y \in \mathfrak{X}(M^n) \quad , \quad N \in \mathfrak{X}^{\perp}(M^n),$$

where $\langle B(X, Y), N\rangle = \langle A^N(X), Y\rangle$ and $D_X N$ is the covariant derivative of the normal connection. A and B are called the second fundamental form of M^n.

The curvature tensors associated with $\bar{\nabla}$, ∇ and D are defined by the following:

$$(3.3) \qquad \begin{aligned} \bar{R}(X, Y) &= [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]} \\ R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \\ R^{\perp}(X, Y) &= [D_X, D_Y] - D_{[X,Y]}. \end{aligned}$$

If the curvature tensor $R^{\perp}$ of the normal connection D vanishes identically, then the normal connection D is said to be flat.

The Gauss equation is given by

$$(3.4)\qquad \langle \bar{R}(Z, Y)X, W\rangle = \langle R(Z, Y)X, W\rangle - \langle B(Y, X), B(Z, W)\rangle + \langle B(X, Z), B(Y, W)\rangle, \quad W, X, Y, Z \in \mathscr{X}(M^n).$$

Moreover we know the following equation:

$$(3.5)\qquad (R(Z, Y)N)^{\perp} = R^{\perp}(Z, Y)N - B(A^N(Y), Z) + B(A^N(Z), Y), \quad Y, Z \in \mathscr{X}(M^n), \quad N \in \mathscr{X}^{\perp}(M^n),$$

where $(R(Z, Y)N)^{\perp}$ is the normal projection of $\bar{R}(Z, Y)N$.

Let M^n be a submanifold immersed in a $2(n+p)$-dimensional almost Hermitian manifold $\bar{M}^{2(n+p)}$ $(p \geq 0)$ with almost complex structure J. We call M^n a totally real (anti-holomorphic) submanifold of $\bar{M}^{2(n+p)}$ if M^n admits an isometric immersion into $\bar{M}^{2(n+p)}$ such that

$$J(T_m(M^n)) \subset \nu_m,$$

where $T_m(M^n)$ denotes the tangent space of M^n at m and ν_m the normal space at m.

§ 4. PROOF OF THEOREMS

Let M^n be a totally real submanifold in an almost Tachibana manifold $\bar{M}^{2n}$ with almost complex structure J.

Let $E_1, E_2 \cdots, E_n$ be othonormal basis of $\mathscr{X}(M^n)$, then by the definition of totally real submanifold, $\mathscr{X}^{\perp}(M^n)$ is spanned by $JE_1, \cdots, JE_n$. Therefore, if $N \in \mathscr{X}^{\perp}(M^n)$, then $JN \in (M^n)$.

PROPOSITION. *Let* M^n *be a totally real submanifold of an almost Tachibana manifold* $\bar{M}^{2n}$ *with almost complex structure* J. *Then we have*

$$(4.1)\qquad A^{JY}(Y) = A^{JY}(X), \quad X, Y \in \mathscr{X}(M^n).$$

Proof. Since $JX \in \mathscr{X}^{\perp}(M^n)$, it follows that

$$(4.2)\qquad \bar{\nabla}_Y JX = -A^{JX}(Y) + D_Y JX$$

by virtue of (3.2). On the other hand, using (3.1), we get

$$(4.3)\qquad \bar{\nabla}_Y JX = (\bar{\nabla}_Y J)X + J\nabla_Y X + JB(X, Y)$$

and hence, regarding with (4.2), it holds that

$$-A^{JX}(Y) - A^{JY}(X) + D_Y JX + D_X JY = J(\nabla_Y X + \nabla_X Y + 2B(X, Y)).$$

Comparing with the tangent part of this, we have

$$2\langle B(X, Y), JZ\rangle = \langle A^{JX}(Y), Z\rangle + \langle A^{JY}(X), Z\rangle$$

for $Z \in \mathscr{X}(M^n)$. If we subtract the equation obtained by interchanging the vectors X and Z in this equation, then we can obtain

$$\langle B(X, Y), JZ\rangle = \langle B(X, Z), JY\rangle,$$

which means (4.1)

Now, we shall show the Theorem 1 and 2 stated in 1.

Proof of Theorem 1. From (3.5), making use of (4.1). We have

$$\langle \bar{R}(Z, Y) JX, JW\rangle = \langle R^{\perp}(Z, Y) JX, JW\rangle - \langle B(X, Y), B(W, Z)\rangle$$
$$+ \langle B(X, Z), B(W, Y)\rangle, \qquad W \in \mathscr{X}(M^n).$$

Therefore, by virtue of this and (3.4), we have

$$\bar{R}(Z, Y) JX - J\bar{R}(Z, Y) X + JR(Z, Y) X = R^{\perp}(Z, Y) JX, \tag{4.4}$$

which means that Theorem 1 is proved.

Proof of Theorem 2. Let $\bar{M}^6$ be a special almost Tachibana manifold with constant α. Then, owing to (2.1) and (2.2), the equation (4.4) can be rewritten as follows

$$\langle R(Z, Y) X, W\rangle - \alpha(\langle Z, W\rangle\langle Y, X\rangle - \langle Z, X\rangle\langle Y, W\rangle)$$
$$= \langle R^{\perp}(Z, Y) JX, JW\rangle, \qquad W \in X(M^n),$$

which completes the proof of Theorem 2.

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