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Compactness gauges and fixed points

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Analisi funzionale. — *Compactness gauges and fixed points.* Nota (*) di FRANCESCO S. DE BLASI, presentata dal Socio G. SANSONE.

RIASSUNTO. — In un recente lavoro Jones introduce una nozione di misura di compattezza e, servendosi di questa, prova un teorema generale di punto fisso. Nella presente Nota si propone una nozione di misura di compattezza contenente in parte quella di Jones. Si dimostra quindi un teorema di punto fisso contenente oltre ai teoremi di Banach, Schauder e Darbo anche il teorema di Sadovskiĭ (non incluso nel teorema di Jones).

1. INTRODUCTION

In a recent paper Jones [8] has proven a general fixed point theorem which contains as special cases both the Banach contraction principle and the Schauder fixed point theorem. His theorem includes also an important generalization of the Schauder theorem due to Darbo [2]. To this end he introduces several notions, among them that of *reducible mapping*, of *p-closure* and, what is probably the most useful, the notion of *compactness gauge* which is a natural generalization of the Kuratowski functional α [12, p. 412]. Furthermore he shows how to construct (non trivial) compactness gauges.

The aim of this paper is to develop Jones' theory in a different direction. We retain his definitions of reducible mapping and p-closure but we use a definition of compactness gauge which differs from that of [8]. Though both contain the definition of the Kuratowski functional α, actually neither implies the other. Constructive methods for obtaining compactness gauges in our different setting are also considered. With the help of our notion of compactness gauge we prove a fixed point theorem which extends in part that of Jones and contains, besides the aforementioned fixed point theorems, also a recent generalization of Darbo's theorem due to Sadovskiĭ [14] (which was not included in the theorem of Jones).

In conclusion we observe that the general theory developed in [8], of which the present paper is in a certain sense a continuation, seems to offer a wide range of applications in the analysis of partial differential equations and functional differential and integral equations of various kinds (see Hale and Cruz [6], Jones [7], [9], [10]).

2. NOTATION AND BASIC DEFINITIONS

Denote by: (S, d) a complete metric space; 2^S the set of all non void subsets of S; C(S) (resp. B(S), K(S)) the set of all non void closed (resp. bounded, compact) subsets of S; conv (S) the set of all non void compact convex

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subsets $A \in S$, where S is a non void bounded closed convex subset of a Banach space; $Z = 0, 1, 2, \cdots$, $R^+ = [0, +\infty)$, $\bar{R}^+ = R^+ \cup \{+\infty\}$; $r(A)$, $A \in 2^S$, the diameter of A. The closure of a set $A \subseteq S$ is denoted by $\bar{A}$. If A is a non void subset of a Banach space, $\overline{co}A$ denotes the closed convex hull of A. For any $A \in 2^S$, $\varepsilon > 0$, set $V(A, \varepsilon) = \{x \in S : d(x, y) < \varepsilon$ for some $y \in A\}$. Let $f : S \to S$. Any non void set $A \subseteq S$ satisfying $fA \subseteq A$ is said to be invariant under f. For any $x \in S$, the set $\bigcup_{n=0}^{\infty} \{f^n x\}$ $(f^0 x = x, f^{n+1} x = ff^n x, n \in Z)$ is called the *orbit* of f generated by x.

DEFINITION 1. *Let* N(S) *be a non void subset of* K(S). *A functional* $p : 2^S \to \bar{R}^+$ *is called a compactness gauge on* 2^S *(with null set* N(S)) *if the following properties are satisfied*: (i) *for any* $\{a\} \in 2^S$ *such that* $p\{a\} = 0$ *and any* $A \in 2^S$, *we have* $p(\{a\} \cup A) = pA$, (ii) $A \subseteq B$ *implies* $pA \le pB$, (iii) $pA = p\bar{A}$, (iv) $pA = 0$ *is equivalent to* $\bar{A} \in N(S)$, (v) *the restriction of* p *to* B(S) *takes values from* R^+.

DEFINITION 2. *A compactness gauge* p *is said to be continuous if it satisfies* $pN(A, \varepsilon) \le pA + s(\varepsilon, pA)$, *for any* $A \in B(S)$ *and* $\varepsilon > 0$, *where* $s : R^+ \times R^+ \to R^+$ *is such that for each* $\eta > 0$, $\lim_{\varepsilon \to 0} s(\varepsilon, \eta) = 0$. *When* s *depends only on* ε, p *is said to be uniformly continuous.*

DEFINITION 3 [8]. *Let* N(S) *be a non void subset of* K(S). *A functional* $p : 2^S \to \bar{R}^+$ *is called a compactness gauge (in the sense of Jones) if it satisfies, in addition to conditions* (ii)–(v) *of Definition* 1, *the following two*: (vi) N(S) *is such that if* $A \in K(S)$ *and, for every* $\varepsilon > 0$ *there exists* $B \in N(S)$ *such that* $A \subseteq V(B, \varepsilon)$, *then* $A \in N(S)$, (vii) if $\{A_n\}_{n=1}^{\infty}$, $A_n \in 2^S$, *satisfies* $A_1 \supseteq A_2 \supseteq \cdots$ *and* $\lim_{n \to \infty} pA_n = 0$, *then for each* $\varepsilon > 0$ *there exist* $B \in K(S)$ *and a positive integer* n *such that* $A_n \subseteq V(B, \varepsilon)$.

The Kuratowski functional α is a (uniformly continuous) compactness gauge in the meaning of Definitions 1 and 3. However the following two examples show that neither definition is stronger than the other.

Example 1. Let S be a Banach space, $B(x_0, a) = \{x \in S : \|x - x_0\| \le a\}$, $x_0 \in S$, $a > 0$. Let N(S) be the set of all non void compact subsets of B, $B = \overline{B(0, 1)}$. For any $A \in 2^S$, define

$$pA \begin{cases} = 0 & \text{if } \bar{A} \subseteq B \text{ is compact,} \\ = 1/2 & \text{if } \bar{A} \subseteq B \text{ is not compact,} \\ = \exp[-D(\bar{A} \setminus B, B)] & \text{if } \bar{A} \nsubseteq B \text{ and } \bar{A} \cap B \text{ is compact,} \\ = 1/2 + \exp[-D(\bar{A} \setminus B, B)] & \text{if } \bar{A} \nsubseteq B \text{ and } \bar{A} \cap B \text{ is not compact,} \end{cases}$$

where $D(A_1, A_2) = \inf\{\|a_1 - a_2\| : a_1 \in A_1, a_2 \in A_2\}$. It is easy to verify that p is a compactness gauge in the sense of Definition 1 but, however, not in the sense of Jones. For, if $A_n = S \setminus B(0, n+1)$, $n \in Z$, we have $A_1 \supseteq A_2 \supseteq \cdots$, $\lim_{n\to\infty} pA_n = 0$ but condition (vii) of Definition 3 is not fulfilled.

Example 2. Keep on the notation of Example 1. Let $x_0 \in S$, $\|x_0\| = 1$. For any $A \in 2^S$ define

$$pA \begin{cases} = \alpha\bar{A} \quad \text{if } \bar{A} \subseteq B, \\ = \alpha(\bar{A} \cap B) + \sup\{\|x - x_0\| : x \in A\} \quad \text{if } \bar{A} \text{ is bounded and} \\ \qquad \bar{A} \cap B,\ \bar{A} \setminus B \neq \varnothing, \\ = \sup\{\|x - x_0\| : x \in A\} \quad \text{if } \bar{A} \text{ is bounded and } \bar{A} \cap B = \varnothing, \\ = +\infty \quad \text{if } \bar{A} \text{ is unbounded,} \end{cases}$$

where α is the Kuratowski functional. It is easy to verify that p is a compactness gauge in the sense of Jones with null set $N(S)$ consisting of all non void compact subsets of B. But p does not satisfy Definition 1 for we have $1/2 = pA < p(\{0\} \cup A) = 1$, where $A = \{tx_0 : 1 \leq t \leq 3/2\}$.

DEFINITION 4. *Let* S *be bounded. A compactness gauge p is said to be weakly contractive on* 2^S *for* $F: 2^S \to 2^S$ *if for any* $A \in 2^S$ *such that* $pA > 0$ *we have* $pFA < pA$. *When, for any* $A \in 2^S$, *we have* $pFA \leq hpA$, $0 \leq h < 1$, *p is said to be contractive on* 2^S *for* F.

Remark 1. Contractive compactness gauges have been studied by Jones in [8]. Special cases of contractive or weakly contractive compactness gauges are considered in [5], [13] (with $p = \alpha$) and [14]. In most applications, if $f: S \to S$ is given, the mapping F is defined by $FA = f^k A$, $A \in 2^S$, $k \geq 1$ a fixed integer.

If f is completely continuous, the Kuratowski functional α is contractive for f.

DEFINITION 5 [8]. *Let* $f: S \to S$. *Let* W *be a non void subset of* 2^S. *f is said to be reducible on* W *if any* $A \in W$, *with* $r(A) > 0$, *which is invariant under f, contains a proper subset* $B \in W$ *also invariant under f.*

Remark 2. Any contraction (or weak contraction) i.e. any function $f: S \to S$ satisfying $d(fx, fy) \leq hd(x, y)$, $x, y \in S$, $0 \leq k < 1$ (or $d(fx, fy) < d(x, y)$, $x, y \in S$, $x \neq y$) is reducible on $K(S)$. Moreover, if S is a non void bounded closed convex subset of a Banach space, any continuous function is reducible on conv (S) [3, p. 454).

DEFINITION 6 [8]. *Let p be any compactness gauge on* 2^S. *A mapping* $h: 2^S \to C(S)$ *is called a p-closure on* 2^S *if satisfies*: (i) $h^2 = h$, (ii) $A \subseteq hA$, (iii) $phA = pA$, (iv) $B \subseteq A$ *implies* $hB \subseteq hA$.

Remark 3. If h is the closure operator in S and $p = \alpha$, all conditions of Definition 6 are fulfilled. The same happens if S is a Banach space and $h = \overline{\text{co}}$. Observe that $N(S)$ is invariant under h, i.e. $hN(S) \subseteq N(S)$.

3. CONSTRUCTION OF COMPACTNESS GAUGES

A method for constructing compactness gauges in the sense of Jones is presented in [8]. The following theorem shows that the method of [8] can be adapted so as to furnish a compactness gauge in the sense of Definition 1.

THEOREM 1. *Let* $m : \mathrm{R}^+ \to \mathrm{R}^+$ *and* $\mathrm{M} : \mathrm{R}^+ \to \mathrm{R}^+$ *be continuous functions such that* $m(0) = \mathrm{M}(0) = 0$, $m(s) \leq \mathrm{M}(s)$, $s > 0$ *and, for each* $s_0 > 0$ *let there exist* $b_0 > 0$ *such that* $m(s) \geq b_0$ *for all* $s \geq s_0$. *Let* $v : \mathrm{S} \times \mathrm{S} \to \mathrm{R}^+$ *be a continuous function such that* $v(x, x) = 0$ *and* $m(\mathrm{d}(x, y)) \leq v(x, y) \leq \leq \mathrm{M}(\mathrm{d}(x, y))$. Let $\mathrm{N}(\mathrm{S})$ *be a non void subset of* $\mathrm{K}(\mathrm{S})$ *satisfying condition* (vi) *of Definition* 3 *and such that, if* $\mathrm{B}_1, \mathrm{B}_2 \in \mathrm{N}(\mathrm{S})$, *also* $\mathrm{B}_1 \cup \mathrm{B}_2 \in \mathrm{N}(\mathrm{S})$. *For each* $a \in \mathrm{S}$, $\mathrm{A} \in 2^{\mathrm{S}}$ *and* $\mathrm{B} \in \mathrm{N}(\mathrm{S})$ *let* $v(a, \mathrm{B}) = \inf \{v(a, b) : b \in \mathrm{B}\}$, $q(\mathrm{A}, \mathrm{B}) = \sup \{v(a, \mathrm{B}) : a \in \mathrm{A}\}$. *Then the functional* $p : 2^{\mathrm{S}} \to \overline{\mathrm{R}}^+$ *defined by* $p\mathrm{A} = \inf \{q(\mathrm{A} : \mathrm{B}) : \mathrm{B} \in \mathrm{N}(\mathrm{S})\}$, $\mathrm{A} \in 2^{\mathrm{S}}$, *is a compactness gauge on* 2^{S}.

Proof. It is shown in [8] that p is a compactness gauge in the sense of Jones. Thus, to prove the theorem we need only to check that condition (i) of Definition 1 is fulfilled. We shall prove a little more, namely that p satisfies

$$p(\mathrm{A}_1 \cup \mathrm{A}_2) = \max \{p\mathrm{A}_1, p\mathrm{A}_2\}, \qquad \mathrm{A}_1, \mathrm{A}_2 \in 2^{\mathrm{S}}.$$

If $\max \{p\mathrm{A}_1, p\mathrm{A}_2\} = +\infty$ there is nothing left to prove. Therefore suppose $p\mathrm{A}_1, p\mathrm{A}_2 < +\infty$. Let $\varepsilon > 0$. From the definition of p, there exist $\mathrm{B}_1, \mathrm{B}_2 \in \mathrm{N}(\mathrm{S})$ such that

$$p\mathrm{A}_i \leq q(\mathrm{A}_i, \mathrm{B}_i) < p\mathrm{A}_i + \varepsilon, \qquad (i = 1, 2).$$

We claim that

$$q(\mathrm{A}_1 \cup \mathrm{A}_2, \mathrm{B}_1 \cup \mathrm{B}_2) \leq \max \{q(\mathrm{A}_1, \mathrm{B}_1), q(\mathrm{A}_2, \mathrm{B}_2)\}.$$

Let $\varepsilon_1 > 0$. From the definition of q, there exists $a_1 \in \mathrm{A}_1 \cup \mathrm{A}_2$, say $a_1 \in \mathrm{A}_1$, such that $q(\mathrm{A}_1 \cup \mathrm{A}_2, \mathrm{B}_1 \cup \mathrm{B}_2) < v(a_1, \mathrm{B}_1 \cup \mathrm{B}_2) + \varepsilon_1$. Thus, since $v(a_1, \mathrm{B}_1 \cup \mathrm{B}_2) \leq v(a_1, \mathrm{B}_1) \leq q(\mathrm{A}_1, \mathrm{B}_1)$, we have $q(\mathrm{A}_1 \cup \mathrm{A}_2, \mathrm{B}_1 \cup \mathrm{B}_2) \leq \leq q(\mathrm{A}_1, \mathrm{B}_1) + \varepsilon_1$. Since a similar inequality holds in case $a_1 \in \mathrm{A}_2$, the claim is true. We remember that $\mathrm{B}_1, \mathrm{B}_2 \in \mathrm{N}(\mathrm{S})$ implies $\mathrm{B}_1 \cup \mathrm{B}_2 \in \mathrm{N}(\mathrm{S})$. Hence

$$\begin{aligned} p(\mathrm{A}_1 \cup \mathrm{A}_2) &\leq q(\mathrm{A}_1 \cup \mathrm{A}_2, \mathrm{B}_1 \cup \mathrm{B}_2) \leq \\ &\leq \max \{q(\mathrm{A}_1, \mathrm{B}_1), q(\mathrm{A}_2, \mathrm{B}_2)\} < \\ &< \max \{p\mathrm{A}_1, p\mathrm{A}_2\} + \varepsilon, \end{aligned}$$

which yields $p(\mathrm{A}_1 \cup \mathrm{A}_2) \leq \max \{p\mathrm{A}_1, p\mathrm{A}_2\}$. The reverse inequality is obvious. This completes the proof.

In the following theorem we are concerned with the construction of a (uniformly) continuous compactness gauge.

THEOREM 2. *Let* $v : S \times S \to R^+$ *satisfy* $v(x, y) \leq v(x, z) + v(z, y)$, $x, y, z \in S$ *and*, $v(x, y) \leq s(d(x, y))$, $x, y \in S$, *where* $s : R^+ \to R^+$, $s(0) = 0$, *is continuous and increasing. Then, if* p *is defined as in Theorem* 1, *for any* $A \in 2^S$ *and* $\varepsilon > 0$ *we have* $pV(A, \varepsilon) \leq pA + s(\varepsilon)$. *Moreover, if all hypotheses of Theorem* 1 *are satisfied,* p *is a uniformly continuous compactness gauge on* 2^S.

Proof. Let $A \in 2^S$, $\varepsilon > 0$. Let $x \in V(A, \varepsilon)$, $B \in N(S)$. If $a \in A$ satisfies $d(a, x) < \varepsilon$ and $b \in B$, we have $v(x, b) \leq v(x, a) + v(a, b) \leq s(d(x, a)) + v(a, b)$ from which, $v(x, B) \leq v(a, B) + s(\varepsilon)$. This inequality implies $q(V(A, \varepsilon), B) \leq q(A, B) + s(\varepsilon)$ which furnishes $pV(A, \varepsilon) \leq pA + s(\varepsilon)$. The last statement is obvious.

4. AUXILIARY RESULTS

In this paragraph we establish a number of results that will be used to prove our fixed point theorem. We start with the following

LEMMA 1. *Let* p *be any compactness gauge on* 2^S. *Let* $A \in N(S)$. *Then any non void compact set* $B \subseteq A$ *is in* $N(S)$. *In particular, for any* $a \in A$ *we have* $p\{a\} = 0$.

Proof. From Definition 1.

LEMMA 2. *Let* $\{A_i\}_{i \in I}$, $A_i \in 2^S$, *be such that* $A_i = hA_i$ *and* $A = \bigcap_{i \in I} A_i \neq \varnothing$. *Then* $A = hA$.

Proof. Trivially $A \subseteq hA$. On the other hand

$$A = \bigcap_{i \in I} A_i = \bigcap_{i \in I} hA_i \supseteq h \bigcap_{i \in I} A_i = hA .$$

Hence $A = hA$.

THEOREM 3. *Let* $f : S \to S$ *be given. Let* p, h *be a compactness gauge and a* p*-closure on* 2^S *and suppose that the set* $W = \{B \in hN(S) : fB \subseteq B\}$ *is non void. Then there exists in* W *a minimal element* S^*. *If* f *is continuous and* h *denotes the closure operator,* $S^* = fS^*$. *Moreover, if* f *is reducible on* $hN(S)$, f *has at least one fixed point.*

Proof. Introduce in W the partial ordering of the set inclusion and let $\{B_i\}_{i \in I}$ be any completely ordered subset in W. We claim that $B = \bigcap_{i \in I} B_i$ is in W, i.e. $B \in hN(S)$ and $fB \subseteq B$. B is non void and compact and, by Lemma 1, $B \in N(S)$ for $B \subseteq B_i$ where $B_i \in N(S)$. Since $B_i = hB_i$, by Lemma 2, we get $B = hB$. Thus $B \in hN(S)$. Finally

$$fB = f \bigcap_{i \in I} B_i \subseteq \bigcap_{i \in I} fB_i \subseteq \bigcap_{i \in I} B_i = B$$

shows that B is invariant under f and $B \in W$. Clearly B is a lower bound for $\{B_i\}_{i \in I}$. Zorn's lemma yields the existence of some minimal element $S^* \in W$. If f is continuous and h the closure operator $fS^* = S^*$ for, otherwise, fS^*

would be a proper compact subset of S^* invariant under f, in contradiction to the minimality of S^*. If f is reducible on $h\mathrm{N}(S)$ we have, by the minimality of $S^*, r(S^*) = 0$ i.e. $S^* = \{a\}$, thus $a = fa$. This completes the proof.

The next theorem develops an idea due to Martelli [13].

THEOREM 4. *Let $f : S \to S$ be continuous and suppose that there exists a set $S^* \in K(S)$ such that $fS^* = S^*$. Let p, h be a compactness gauge and a p-closure on 2^S. Let $S = hS$. Then there exists a closed set $A^* \supseteq S^*$ such that $hfA^* = A^*$ and $A^* = hA^*$.*

Proof. Let $W = \{B \in C(S) : B = hB , fB \subseteq B , B \text{ and } fB \supseteq S^*\}$. W is non void since $S \in W$. Define.

$$A^* = \bigcap_{B \in W} B.$$

A^* contains S^*, hence it is non void. Furthermore, by Lemma 2, $A^* = hA^*$. Let $B \in W$. Then $fB \subseteq B$ implies $hfB \subseteq hB = B$. Thus, we obtain

$$hfA^* = hf \bigcap_{B \in W} B \subseteq \bigcap_{B \in W} hfB \subseteq \bigcap_{B \in W} B = A^*.$$

We claim that the last inclusion is actually an equality. Indeed, $B_1 = hfA^*$ is in $C(S)$ and satisfies both $hB_1 = B_1$ and $B_1 , fB_1 \supseteq S^*$. Since

$$fB_1 = fhfA^* \subseteq fA^* \subseteq hfA^* = B_1 ,$$

B_1 is invariant under f. Therefore $B_1 \in W$ and $B_1 = hfA^* \supseteq A^*$. From this and $hfA^* \subseteq A^*$ the equality follows and the proof is complete.

5. A FIXED POINT THEOREM

Using the results established in the preceding paragraph we can now prove the following fixed point theorem.

THEOREM 5. *Let $f : S \to S$, where S is a bounded complete metric space, be continuous. Let p, h be a compactness gauge and a p-closure on 2^S and suppose $S = hS$. Let p be weakly contractive on 2^S for f^k ($k \geq 1$ a fixed integer) and, if $k \geq 2$, let the null set $\mathrm{N}(S)$ of f contain all one point sets. Let f be reducible on $h\mathrm{N}(S)$. Then f has at least one fixed point.*

Proof. Consider the case $k = 1$ (the proof for $k \geq 2$ is similar). By Lemma 1, choose $a \in S$ such that $p\{a\} = 0$ and let $A = \bigcup_{n=0}^{\infty} \{f^n a\}$. From $A = \{a\} \cup fA$ we obtain $pA = pfA$ which implies $pA = 0$. Thus $\bar{A} \in \mathrm{N}(S)$. Define

$$W_0 = \{B \in \mathrm{N}(S) : fB \subseteq B\} .$$

Since f is continuous, $f\bar{A} \subseteq \bar{A}$ and W_0 is non void. From Theorem 3, being f continuous, there exists $S^* \in W_0$ such that $fS^* = S^*$. By hypothesis $S = hS$. Therefore all assumptions of Theorem 4 are satisfied and there exists a set $A^* \in C(S)$ such that

$$hfA^* = A^* \quad \text{and} \quad A^* = hA^*.$$

Then

$$phfA^* = pfA^* = pA^*,$$

furnishes $pA^* = 0$ and, since $A^* = hA^*$, $A^* \in hN(S)$. Thus the set W defined in Theorem 3 is non void (for $A^* \in W$) and, being f reducible on $hN(S)$, by Theorem 3 f has at least one fixed point. The proof is complete.

Remark 4. The conclusion of Theorem 5 remains true if the hypothesis that f is reducible on $hN(S)$ is replaced by any of the following two: (i) for any $A \in N(S)$, hA is one-point set or, (ii) f has the fixed point property on $hN(S)$.

Remark 5. Theorem 5 contains the fixed point theorems of Schauder and Banach with a number of their generalizations due to Dardo [2], Sadovskiĭ [14], Edelstein [4], Browder [1], Krasnosel'skiĭ and Stecenko [11], Furi and Vignoli [5]. Theorem 5 extends in part a theorem of Jones [8].

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