Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

H. M. Srivastava

Some finite summation formulas

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.3-4, p. 166–169.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_3-4_166_0>

Analisi matematica. — *Some finite summation formulas* (*). Nota (**)
di HARI M. SRIVASTAVA, presentata dal Socio G. SANSONE.

RIASSUNTO. — L'Autore considera una successione di funzioni $\{ H_n^{(\lambda)}(x) \}$ generata dal prodotto di due serie di potenze di cui una è e^z oppure $e_q[z]$ e trova per le $H_n^{(\lambda)}(x)$ alcune formule che in particolare ne comprendono altre da lui ottenute is precedenza.

1. INTRODUCTION

Put

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad , \quad G[z] = \sum_{n=0}^{\infty} g_n z^n \, , \quad g_n \neq 0 \, . \tag{1}$$

In the present Note we first prove a summation formula given by

THEOREM 1. *Let* $E(z)$ *and* $G[z]$ *be defined by* (1), *and let the sequence of functions* $\{H_n^{(\lambda)}(x) \mid n = 0, 1, 2, \cdots\}$ *be generated by*

$$E(M_1(x)\, t^{m_1} + \cdots + M_r(x)\, t^{m_r})\, G[Q(x)\, t^p] = \sum_{n=0}^{\infty} \frac{t^n}{(\lambda+1)_n} H_n^{(\lambda)}(x), \tag{2}$$

where $M_j(x)\, , j = 1 \, , \cdots , r$, *and* $Q(x) \neq 0$ *are real functions*, $m_1 \, , \cdots , m_r \, , p$ *are positive integers*, *and* λ *is an arbitrary complex number.*

Then

$$H_n^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{n/p} \sum_{k_1,\cdots,k_r=0}^{m_1k_1+\cdots+m_rk_r \leq n} \binom{\lambda+n}{m_1k_1+\cdots+m_rk_r} (m_1k_1+\cdots+m_rk_r)! \cdot \tag{3}$$

$$\cdot \frac{[\Delta_{m_1/p}(M_1 Q)]^{k_1}}{k_1!} \cdots \frac{[\Delta_{m_r/p}(M_r Q)]^{k_r}}{k_r!} H^{(\lambda)}_{m-m_1k_1-\cdots-m_rk_r}(y) \, ,$$

where, for convenience,

$$\Delta_\nu(UV) = U(x) \left[\frac{V(y)}{V(x)}\right]^\nu - U(y) \, , \tag{4}$$

for all ν *and any given pair of functions* U *and* V.

2. PROOF OF THEOREM 1

In order to prove the summation formula (3), we start from the generating function (2) and set $t = [Q(y)]^{1/p} z$. We thus have

$$E\left(\sum_{j=1}^{r} M_j(x)\, [Q(y)]^{m_j/p} z^{m_j}\right) G\,[Q(x)\, Q(y)\, z^p] = \sum_{n=0}^{\infty} \frac{[Q(y)]^{n/p}}{(\lambda+1)_n} H_n^{(\lambda)}(x)\, z^n. \tag{5}$$

(*) Supported in part by NRC grant A-7353.
(**) Pervenuta all'Accademia il 10 settembre 1974.

which, on interchanging x and y, yields

(6) $$E\left(\sum_{j=1}^{r} M_j(y)\,[Q(x)]^{m_j/p} z^{m_j}\right) G\,[Q(x)\,Q(y)\,z^p] = \sum_{n=0}^{\infty} \frac{[Q(x)]^{n/p}}{(\lambda+1)_n}\, H_n^{(\lambda)}(y)\, z^n.$$

If we compare (5) and (6), we get

$$\sum_{n=0}^{\infty} \frac{[Q(y)]^{n/p}}{(\lambda+1)_n}\, H_n^{(\lambda)}(x)\, z^n = E\left(\sum_{j=1}^{r} [M_j(x)\,[Q(y)]^{m_j/p} - M_j(y)\,[Q(x)]^{m_j/p}]\, z^{m_j}\right) \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{[Q(x)]^{n/p}}{(\lambda+1)_n}\, H_n^{(\lambda)}(y)\, z^n = \prod_{j=1}^{r} E\,(\Delta_{m_j/p}(M_j Q)\,[Q(x)]^{m_j/p} z^{m_j}) \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{[Q(x)]^{n/p}}{(\lambda+1)_n}\, H_n^{(\lambda)}(y)\, z^n = \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \frac{[\Delta_{m_1/p}(M_1 Q)]^{k_1}}{k_1!} \cdots \frac{[\Delta_{m_r/p}(M_r Q)]^{k_r}}{k_r!} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{[Q(x)]^{(n+m_1k_1+\cdots+m_rk_r)/p}}{(\lambda+1)_n}\, H_n^{(\lambda)}(y)\, z^{n+m_1k_1+\cdots+m_rk_r},$$

and on replacing n by $n - m_1 k_1 - \cdots - m_r k_r$, we readily obtain

$$\sum_{n=0}^{\infty} \frac{[Q(y)]^{n/p}}{(\lambda+1)_n}\, H_n^{(\lambda)}(x)\, z^n = \sum_{n=0}^{\infty} [Q(x)]^{n/p} z^n \cdot$$

$$\cdot \sum_{k_1,\cdots,k_r=0}^{m_1k_1+\cdots+m_rk_r \leq n} \frac{[\Delta_{m_1/p}(M_1 Q)]^{k_1}}{k_1!} \cdots \frac{[\Delta_{m_r/p}(M_r Q)]^{k_r}}{k_r!} \cdot$$

$$\cdot \{(\lambda+1)_{n-m_1k_1-\cdots-m_rk_r}\}^{-1} H_{n-m_1k_1-\cdots-m_rk_r}^{(\lambda)}(y).$$

Now equate the coefficients of z^n on either side of the last equation, and the summation formula (3) would follow at once.

This evidently completes the proof of Theorem 1.

3. PARTICULAR CASES

To derive the special case of Theorem 1 when $r = 1$, we may set $M_j(x) = 0$, $j = 2, \cdots, r$, and then drop the subscript 1 in the resulting forms of equations (2) and (3). We thus obtain

COROLLARY 1. *Let* $E(z)$ *and* $G[z]$ *be defined by* (1), *and let*

(7) $$E(M(x)\,t^m)\,G\,[Q(x)\,t^p] = \sum_{n=0}^{\infty} \frac{t^n}{(\lambda+1)_n}\, \Psi_n^{(\lambda)}(x),$$

where $M(x) \neq 0$, $Q(x) \neq 0$ *are real functions,* m, p *are positive integers, and* λ *is an arbitrary constant, real or complex.*

Then

(8) $$\Psi_{mn}^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{mn/p} \sum_{k=0}^{n} \binom{\lambda+mn}{mk} \frac{(mk)!}{k!} \cdot [\Delta_{m/p}(MQ)]^k\, \Psi_{mn-mk}^{(\lambda)}(y)$$

and

$$\Psi_{mn}^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{mn/p} \sum_{k=0}^{n} \binom{\lambda+mn}{mn-mk} \frac{(mn-mk)!}{(n-k)!} \cdot [\Delta_{m/p}(MQ)]^{n-k}\, \Psi_{mk}^{(\lambda)}(y). \tag{9}$$

Evidently the last formula (9) can be deduced by reversing the order of summation in (8).

Next we consider the special case of Theorem 1 when $r = 2$. In equations (2) and (3) if we put $M_j(x) = 0$, $j = 3, \cdots, r$, let $m_1 = m$, $m_2 = l$, and replace $M_1(x)$, $M_2(x)$ by $M(x)$ and $N(x)$, respectively, we are led at once to

COROLLARY 2. *With* $E(z)$ *and* $G[z]$ *defined by* (1), *let*

$$E(M(x)\,t^m + N(x)\,t^l)\, G[Q(x)\,t^p] = \sum_{n=0}^{\infty} \frac{t^n}{(\lambda+1)_n}\, S_n^{(\lambda)}(x), \tag{10}$$

where $M(x)$, $N(x)$, *and* $Q(x) \neq 0$ *are real functions*, l, m, p *are positive integers, and* λ *is an arbitrary constant, real or complex.*

Then

$$S_n^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{n/p} \sum_{k,j=0}^{mk+lj\leq n} \binom{\lambda+n}{mk+lj} (mk+lj)! \cdot \frac{[\Delta_{m/p}(MQ)]^k}{k!} \frac{[\Delta_{l/p}(NQ)]^j}{j!} S_{n-mk-lj}^{(\lambda)}(y). \tag{11}$$

The summation formulas (8), (9) and (11) are contained, respectively, in Theorem 1, p. 64 and Theorem 5, p. 71 of our recent paper [4]. For several special cases of Corollary 1 and a number of interesting properties of the $\Psi_n^{(\lambda)}(x)$, generated by (7), the reader may be referred to §§1, 3 and 4 of [4, pp. 64–70].

4. A q-ANALOGUE OF THEOREM 1

Recalling the familiar notations of Jackson [3] and Hahn [2] let the basic binomial $[a+b]_n$ abbreviate the n-rank product

$$(a+b)(a+qb)(a+q^2 b)\cdots(a+q^{n-1} b), \qquad |q| < 1, \tag{12}$$

so that since

$$[a+b]_n = a^n \left[1 + \frac{b}{a}\right]_n = [a+b]_{n-1}(a+q^{n-1} b), \tag{13}$$

it follows at once that

$$[a+b]_0 = 1 \tag{14}$$

and

$$[a+b]_{-n} = \frac{1}{(a+bq^{-1})\cdots(a+bq^{-n})} = \frac{q^{n(n+1)/2}\, b^{-n}}{[1+aq/b]_n}. \tag{15}$$

Also let [1, p. 6]

$$\left[\begin{matrix}\alpha\\k\end{matrix}\right] = \frac{(1-q^{\alpha})(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)} = \frac{[1-q^{\alpha-k+1}]_k}{[1-q]_k}, \tag{16}$$

and denote the q-exponential function by

$$e_q[z] = \lim_{k\to\infty} \frac{1}{[1-z]_k} = \sum_{n=0}^{\infty} \frac{z^n}{[1-q]_n}; \tag{17}$$

then it is readily seen that

$$\frac{e_q[z]}{e_q[\zeta]} = \sum_{n=0}^{\infty} \frac{[z-\zeta]_n}{[1-q]_n}. \tag{18}$$

The method of proof of Theorem 1 can be followed step by step to establish its q analogue, given by Theorem 2 below, which is indeed a generalization of our earlier Theorem 6, p. 71 in [4]. We, therefore, omit details and content ourselves by stating

THEOREM 2. *Let* $e_q[z]$ *be defined by* (17) *and* G $[z]$ *by* (1). *Also let*

$$e_q[M_{q,1}(x)\,t^{m_1}]\cdots e_q[M_{q,r}(x)\,t^{m_r}]\,G[Q(x)\,t^p] = \sum_{n=0}^{\infty} \frac{t^n}{[1-q^{\lambda+1}]_n}\,H_{q,n}^{(\lambda)}(x), \tag{19}$$

where $M_{q,j}(x)$, $j = 1,\cdots,r$, *depend, in general, on both the base* q *and the argument* x, $|q| < 1$, $Q(x) \neq 0$ *is a real function,* $m_1,\cdots,m_r, p$ *are positive integers, and* λ *is an arbitrary complex number.*

Then

$$H_{q,n}^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{n/p} \sum_{k_1,\cdots,k_r=0}^{m_1k_1+\cdots+m_rm_r\leq n} \left[\begin{matrix}\lambda+n\\m_1k_1+\cdots+m_rk_r\end{matrix}\right]\cdot[1-q]_{m_1k_1+\cdots+m_rk_r}\cdot \tag{20}$$

$$\cdot\frac{[\Delta_{m_1/p}(M_{q,1}Q)]_{k_1}}{[1-q]_{k_1}}\cdots\frac{[\Delta_{m_r/p}(M_{q,r}Q)]_{k_r}}{[1-q]_{k_r}}\,H_{q,n-m_1k_1-\cdots-m_rk_r}^{(\lambda)}(y).$$

Indeed it is easy to observe that, in the limit as $q\to 1$, Theorem 2, with $M_{q,j}(x)$ replaced by $(1-q)\,M_j(x)$, $j=1,\cdots,r$, and $H_{q,n}^{(\lambda)}(x)$ by $(1-q)^n\,H_n^{(\lambda)}(x)$, would lead us to Theorem 1.

REFERENCES

[1] W. HAHN (1949) – *Über Orthogonalpolynome, die q–Differenzengleichungen genügen*, «Math. Nachr.», *2*, 4–34.

[2] W. HAHN (1949) – *Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation.* «Math. Nachr.», *2*, 340–379.

[3] F. H. JACKSON (1904) – *The application of basic numbers to Bessel's and Legendre's functions*, « Proc. London Math. Soc. (2) », *2*, 192–220.

[4] H. M. SRIVASTAVA (1971) – *On q-generating functions and certain formulas of David Zeitlin* « Illinois J. Math. », *15*, 64–72.