Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the solvability of boundary value problems for elliptic-parabolic systems of second order

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.3-4, p. 147–155.

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<http://www.bdim.eu/item?id=RLINA_1974_8_57_3-4_147_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Matematica. — *On the solvability of boundary value problems for elliptic-parabolic systems of second order.* Nota (*) di JULIUS DUFNER, presentata dal Corrisp. G. FICHERA.

RIASSUNTO — Viene considerato un sistema di p equazioni lineari alle derivate parziali del secondo ordine in p funzioni incognite. Si suppone che il sistema sia ellittico–parabolico [cfr. la (2)].

Considerato per esso un problema « ben posto », secondo una teoria sviluppata in precedenza (cfr. Bibliografia), si dà, sotto opportune ipotesi, un teorema di regolarizzazione per la soluzione debole di questo problema.

1. INTRODUCTION

Let us denote by Ω a bounded connected open set in the real n-space R^n. Let the matrices $a^{ij} = (a^{ij}_{kl})$, $b^i = (b^i_{kl})$, $c = (c_{kl})$ $(1 \leq i, j \leq n, 1 \leq k, l \leq p)$ be of $C^2(\bar{\Omega})$, $C^1(\bar{\Omega})$, $C^0(\bar{\Omega})$ (1) respectively and assume $a^{ij}_{kl} = a^{ji}_{kl} = a^{ij}_{lk}$, $b^i_{kl} = b^i_{lk}$ for all i, j, k, l.

We shall consider the following system of p linear second order differential equations

$$(1) \qquad (Lu)_k = -a^{ij}_{kl} \partial_i \partial_j u_l + b^i_{kl} \partial_i u_l + c_{kl} u_l = f_k \qquad \text{in } \Omega,$$

or briefly: $Lu = -a^{ij} \partial_i \partial_j u + b^i \partial_i u + cu = f$ in Ω. Here $u = (u_1, \cdots, u_p)$, $f = (f_1, \cdots, f_p)$, $\partial_i = \frac{\partial}{\partial x_i}$ and the usual summation convention is used.

We shall suppose that the operator L is elliptic-parabolic in the following sense

$$(2) \qquad a^{ij}_{kl}(x) z^k_i z^l_j \geq 0 \qquad \text{for } x \in \Omega, z^k_i \in R.$$

Let us further assume that

$$(3) \qquad (c_{kl} + c_{lk} - b^i_{kl\,|\,i} - a^{ij}_{kl\,|\,ij})(x) z^k z^l \geq q \cdot |z|^2 \qquad \text{for } x \in \Omega, z^k \in R,$$

$|z|^2 = \sum_{k=1}^{p} (z^k)^2$, $b^i_{kl\,|\,i} = \partial_i b^i_{kl}, \cdots, q$ denoting a positive constant.

In [1], boundary value problems for the equation (1) are formulated which are correctly posed in a certain sense discussed there; among other things,

(*) Pervenuta all'Accademia il 27 settembre 1974.

(1) Denote by $C^k(\Omega)$ $(C^k(\bar{\Omega}))$ the set of real-valued functions that are continuous (uniformly continuous) in Ω together with all their first k derivatives. Likewise all function spaces used will be assumed to be real and no distinct notations will be used for product spaces.

the existence and properties of L_2-weak solutions (see Definition 4 below) are shown and the smooth solvability in a certain special case is proved.

In this paper we repeat briefly in section 2 what we mean by a correctly posed boundary value problem for the equation (1) – for details see [1]. In section 3 we formulate our problem, with the aid of an appropriate bilinear form, as variational problem, so that the boundary conditions are split into «given» data (which are always of order zero) and «natural» boundary conditions. We show that the corresponding H_1-weak solution (see Definition 3), being sufficiently smooth, satisfies the boundary conditions in the classical sense. Using results of J. J. Kohn, L. Nirenberg [3], [4] we show in section 4 the existence of smooth solutions, provided that the boundary of Ω is nowhere characteristic for L. In section 5 the uniqueness of L_2-weak solutions is proved for $\dim(\Omega) = 2$, assuming that the boundary is noncharacteristic. This is done with the aid of an elliptic regularization globally defined and regularizing in tangential direction so that the results of [3], [4] again are applicable. Finally a standard argument furnishes a result about «weak equals strong».

It is assumed in the sections 4 and 5 that the boundary $\partial\Omega$ is noncharacteristic, i.e. $\det(a^{ij} n_i n_j) > 0$, $(n_1, \cdots, n_n)$ denoting the outward normal. The case of characteristic boundary has not been investigated intensively as yet; the following, however, may be said: If $\partial\Omega$ consists, besides of noncharacteristic portions, of a finite number of components on which $a^{ij} n_i n_j = 0$, then under certain conditions Theorem 1 of section 4 is still valid (see [1], [3]). The methods used there cannot be applied if characteristic boundary parts of any other type occur, because the special situation $a^{ij} n_i n_j = 0$ (cfr. [1]) is essentially used. – O. A. Oleinik [5] has treated the Dirichlet problem for a single equation (1) with (2), posed first by G. Fichera [2] (being correctly posed in our sense). However, the methods of [5], based on the maximum principle for a single elliptic equation, cannot be applied in our case of systems.

2. Correctly posed boundary value problems

Let us assume that the domain Ω with boundary $\partial\Omega$ can be described with the aid of a function $\varphi \in C^1(R^n)$:

$$\Omega = \{x \in R^n : \varphi(x) > 0\} \quad , \quad \partial\Omega = \{x \in R^n : \varphi(x) = 0\},$$

$|\nabla\varphi(x)| = 1$ for $x \in \partial\Omega$, so that we may define the outward normal by $(n_1, \cdots, n_n) = -\nabla\varphi$ on $\partial\Omega$.

Let A^j $(1 \leq j \leq n)$ be given symmetric $p \times p$-matrices of class $C^1(\partial\Omega)$ satisfying $A^j n_j = 0$ on $\partial\Omega$. Let $\tilde{A}$ be the symmetric matrix of $C^0(\partial\Omega)$ such that

$$\int_{\partial\Omega} (A^j \partial_j u \cdot v + A^j u \cdot \partial_j v + \tilde{A} u \cdot v)\, d\sigma = 0 \qquad \text{for all} \quad u, v \in C^1(\partial\Omega),$$

$d\sigma$ representing elements of volume on $\partial\Omega$. Setting $b = (b^i + a^{ij}_{\;|j})\, n_i$, we assume that each of the matrices $Q = a^{ij} n_i n_j$ and $\tilde{b} = b + \tilde{A}$ are split continuously into two components:

$$Q = Q_+ + Q_- \quad , \quad \tilde{b} = B + C \qquad \text{on} \quad \partial\Omega.$$

Denote by V_p the linear space of real p-tuples, by Q'_+ the transpose of the matrix Q_+ and set – in a fixed point of $\partial\Omega$ – $R(Q_+) = \{z \in V_p : z = Q_+ y, y \in V_p\}$, $N(Q'_+) = \{z \in V_p : Q'_+ z = 0\}$, $R(B' \,|\, N(Q'_-)) = \{z \in V_p : z = B' y, y \in N(Q'_-)\}$. Denote by $+$ $(\oplus)$ the sum (direct sum) of subspaces of V_p. Then we require

$$(E_1) \qquad R(Q_+) \oplus R(Q_-) = R(Q) \qquad \text{on} \quad \partial\Omega$$

$$(E_2) \qquad \{R(Q_+) + R(B' \,|\, N(Q'_-))\} \cap \{R(Q_-) + R(C' \,|\, N(Q'_+))\} = \{0\} \qquad \text{on} \quad \partial\Omega$$

$$(E_3) \qquad \{R(Q_+) + R(C \,|\, N(Q'_-))\} \cap \{R(Q_-) + R(B \,|\, N(Q'_+))\} = \{0\} \qquad \text{on} \quad \partial\Omega$$

$$(U) \qquad z \cdot (C - B)\, z \geq 0 \quad (z \in N(Q'_+)) \qquad \text{on} \quad \partial\Omega$$

$$(D) \qquad R(Q_-) \supset R(A^1) + \cdots + R(A^n) \qquad \text{on} \quad \partial\Omega.$$

DEFINITION 1. *Assume that the operator* L *satisfies* (2), (3). *Suppose that the matrices* $Q_+, Q_-, B, C \in C^0(\partial\Omega)$ *and* $A^j \in C^1(\partial\Omega)$ $(1 \leq j \leq n)$ *satisfy* $(E_1), (E_2), (E_3), (U), (D)$. *If* $P_\pm$ *denotes a pair of projectors so that* $Q_\pm = P_\pm Q$, *then using the notations* $Q^j = a^{ij} n_i$, $Q^j_\pm = P_\pm Q^j$ *we call*

$$(1) \qquad Lu = f \quad \text{in} \quad \Omega,$$

$$(4) \qquad \begin{cases} (a) \quad Q^j_- \partial_j u - A^j \partial_j u - Bu = 0 & \text{on} \quad \partial\Omega \\ (b) \quad Q'_+ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

a correctly posed boundary value problem.

In what follows we only consider such correctly posed problems.

Denote by $L^* = -a^{ij} \partial_i \partial_j + (-b^i - 2a^{ij}_{\;|j})\, \partial_i + (c' - b^i_{\;|i} - a^{ij}_{\;|ij})$ the formaladjoint of L and by $(\, , \,)$ the scalarproduct in $L_2(\Omega)$. Then the following Green's formula holds:

$$(Lu, v) - (u, L^* v) = \int_{\partial\Omega} [-v(Q^j_- \partial_j u - A^j \partial_j u - Bu) + \partial_j v((Q^j_+)' u)] +$$
$$+ [u(Q^j_- \partial_j v + A^j \partial_j v + C' v) - \partial_j u((Q^j_+)' v)]\, d\sigma.$$

Thus, observing that $N(Q'_+) = \bigcap_{j=1}^{n} N((Q^j_+)')$, the following definition is suggested:

DEFINITION 2. *The boundary value problem*

$$(5) \qquad L^* v = g \quad \text{in} \quad \Omega,$$

$$(6) \qquad \begin{cases} (a) \quad Q^j_- \partial_j v + A^j \partial_j v + C' v = 0 & \text{on} \quad \partial\Omega \\ (b) \quad Q'_+ v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

is called the formal adjoint boundary value problem to (1), (4).

It is obvious that the correctness of (1), (4) implies the correctness of (5), (6) and vice versa. We remark that in general the boundary datas of order zero contained in (4) neither are the same as (4 *b*) nor are the same as the zero datas contained in (6). However this *is* true in such points of $\partial\Omega$ which are noncharacteristic for L, i.e. for which det $(a^{ij} n_i n_j) > 0$.

3. H_1-WEAK SOLUTIONS

In order to define weak solutions belonging to $H_1(\Omega)$ we construct an appropriate Green's formula. Let $\varphi \in C^3(R^n)$, $A^1, \cdots, A^n \in C^2(\partial\Omega)$, $B \in C^1(\partial\Omega)$ and let $A^1, \cdots, A^n$ and B be extended into Ω as $C^2(\overline{\Omega})$— and $C^1(\overline{\Omega})$— functions respectively. Then the matrices $\tilde{a}^{ij} = a^{ij} + (A^i \varphi_{|j} - A^j \varphi_{|i})$ and $B^j = - B\varphi_{|j}$ are of class $C^2(\overline{\Omega})$ and $C^1(\overline{\Omega})$ respectively. Integrating by parts and using $\tilde{A} = (A^i \varphi_{|j} - A^j \varphi_{|i})_{|j} n_i$ on $\partial\Omega$ we obtain

$$(7) \qquad (Lu, v) = Q(u, v) - \int_{\partial\Omega} (Q^j \partial_j u - A^j \partial_j u - Bu) \cdot v \, d\sigma ,$$

$$(8) \qquad Q(u, v) = \int_{\Omega} \tilde{a}^{ij} \partial_i u \cdot \partial_j v + [(\tilde{a}^{ij}{}_{|j} + b^i) - B^i] \partial_i u \cdot v - B^i u \cdot \partial_i v + + (c - B^i{}_{|i}) u \cdot v \, dx = \int_{\Omega} \tilde{a}^{ij} \partial_i u \cdot \partial_j v + + \frac{1}{2} [(\tilde{a}^{ij}{}_{|j} + b^i) \partial_i u \cdot v - (\tilde{a}^{ij}{}_{|j} + b^i) u \cdot \partial_i v] + + \frac{1}{2} (2c - b^i{}_{|i} - a^{ij}{}_{|ij}) u \cdot v \, dx + \frac{1}{2} \int_{\partial\Omega} (\tilde{b} - 2B) u \cdot v \, d\sigma .$$

DEFINITION 3. *Denote by* $\overline{B}$ *the closure in* $H_1(\Omega)$ *of*

$$B = \{u \in C^1(\overline{\Omega}) : Q'_+ u = 0 \text{ on } \partial\Omega\}.$$

Then a function $u \in \overline{B}$ *is called an* H_1*-weak solution of the boundary value problem* (1), (4), *if*

$$(9) \qquad Q(u, v) = (f, v) \qquad \textit{for all} \quad v \in \overline{B}.$$

Because L is degenerated elliptic it does not make sense in general to look for solutions in $H_1(\Omega)$. However we shall later make assumptions such that our solutions are smooth enough.

Using the notation $\| u \|_0^2 = (u, u)$, it follows from (2), (3), (U) that $Q(u, u) \geq q \| u \|_0^2$ for $u \in B$. Hence an H_1-weak solution of (1), (4) satisfies the a-priori estimate $q \| u \|_0 \leq \| f \|_0$.

Obviously any classical solution of class $C^1(\overline{\Omega}) \cap C^2(\Omega)$ is an H_1-weak solution. To investigate the converse we first note an easily proved

LEMMA 1. Let $\varphi \in C^1(R^n)$. *Then any function* u *of class* $\overline{B} \cap C^0(\overline{\Omega})$ *satisfies* $Q'_+ u = 0$ *on* $\partial\Omega$.

Now let u be an H1-weak solution of (1), (4) belonging to $C^2(\overline{\Omega})$.

Because $C_0^\infty(\Omega) \subset B$ we have $Lu = f$ in Ω; using Lemma 1 we get $Q'_+ u = 0$ on $\partial\Omega$ and for all $v \in B : \int_{\partial\Omega} (Q^j_- \partial_j u - A^j \partial_j u - Bu) \cdot v \, d\sigma = 0$.

Thus with the above choice of the bilinear form $Q(u, v)$ the parts (4 *a*), (4 *b*) of our boundary conditions (4) appear as «natural» and «given» boundary conditions respectively.

Sufficient conditions under which the natural boundary conditions (4*a*) are satisfied in the classical sense are given in the following

LEMMA 2. *Let us denote by* Λ *the set of points* $x_0 \in \partial\Omega$ *such that for any vector* $z \in N(Q'_+(x_0))$ *there exists a neighborhood* U_0 *of* x_0 *(on* $\partial\Omega$*) and a function* $v \in C^1(U_0)$ *satisfying* $v(x_0) = z$ *and* $Q'_+ v(x) = 0$ *for* $x \in U_0$. *Assume that* $\varphi \in C^1(R^n)$ *and that* Λ *is dense in* $\partial\Omega$. *Let* u, v *be of class* B. *Then*

(i)
$$\int_{\partial\Omega} w(Q^j_- \partial_j u - A^j \partial_j u - Bu) \, d\sigma = 0 \qquad \textit{for all} \quad w \in B$$
$$\textit{implies} \quad Q^j_- \partial_j u - A^j \partial_j u - Bu = 0 \qquad \textit{on} \quad \partial\Omega,$$

(ii)
$$\int_{\partial\Omega} w(Q^j_- \partial_j v + A^j \partial_j v + C'v) \, d\sigma = 0 \qquad \textit{for all} \quad w \in B$$
$$\textit{implies} \quad Q^j_- \partial_j v + A^j \partial_j v + C'v = 0 \qquad \textit{on} \quad \partial\Omega.$$

Remark. The following result (which is easily proved) can be used to check the density of Λ in $\partial\Omega$: Let Γ denote the set of points of $\partial\Omega$ having a neighborhood (on $\partial\Omega$) in which Q_+ is of constant rank. Assume that $\varphi \in C^1(R^n)$ and that $Q_+ \in C^1(\partial\Omega)$. Then $\Gamma \subset \Lambda$.

Proof of Lemma 2. We first localize the problem. Let $\{O_\tau\}$ be a finite open covering of $\partial\Omega$. Then there exists a continuously differentiable mapping $F_\tau = (F^1_\tau, \cdots, F^n_\tau)$ which maps $O_\tau \cap \overline{\Omega}$ onto $P_\tau = \{y_n \leq 0, y_1^2 + \cdots + y_n^2 < R^2\}$ and $O_\tau \cap \partial\Omega$ onto $S_\tau = \{y_n = 0, y_1^2 + \cdots + y_n^2 < R^2\}$. Consider fixed $O_\tau = O$, $F_\tau = F, \cdots$. Let $B(S) = \{u \in C^1_0(S) : Q'_+ u = 0 \text{ on } S\}$, $y' = (y_1, \cdots, y_{n-1})$, $\omega \, dy' = d\sigma$. Then we get

$$\text{(10)} \qquad \int_S w(Q^j_- F^l_{x_j} \partial_{y_l} u - A^j F^l_{x_j} \partial_{y_l} u - Bu) \, \omega \, dy' = 0 \qquad \text{for} \quad w \in B(S).$$

Second, we show that from (10) it follows

$$\text{(11)} \quad z[(Q^j_- F^l_{x_j} \partial_{y_l} u - A^j F^l_{x_j} \partial_{y_l} u - Bu)(y)] = 0 \;\text{ for } z \in N(Q'_+(y)),\; y \in F(\Lambda \cap O).$$

For this end let us assume that $u_0 \in B$, $y_0 \in F(\Lambda \cap O)$, $z_0 \in N(Q'_+(y_0))$ and that $z_0[(Q^j_- F^l_{x_j} \partial_{y_l} u_0 - A^j F^l_{x_j} \partial_{y_l} u_0 - Bu_0)(y_0)] > 0$. Then there exists a neighborhood U_0 of y_0 and a function $w_0 \in C^1(U_0)$ satisfying $w_0(y_0) = z_0$

and $Q'_+ w_0(y) = 0$ for $y \in U_0$. Now cut off w_0 by a suitable nonnegative function $\zeta \in C_0^\infty(S)$ such that $[(\zeta w_0)(Q^j_- F^l_{x_j} \partial_{y_l} u_0 - A^j F^l_{x_j} \partial_{y_l} u_0 - B u_0)](y) \geq 0$ for $y \in S$ – contradiction to (10).

Finally, observing that $R(Q_-) = \sum_{j=1}^{n} R(Q^j_-)$ and by condition (D), we see that the expression in square brackets in (11) is a vector belonging to $N(Q'_+)^\perp \cap \cap \{R(Q_-) + R(B \mid N(Q'_+))\}$ for $y \in F(\Lambda \cap O)$. Hence using condition (E_3) we get $Q^j_- \partial_j u - A^j \partial_j u - Bu = 0$ on $\Lambda \cap O$. Thus (i) is proved by continuity of all functions occurring. By the same argument (ii) is shown, using condition (E_2) instead of (E_3).

4. Smooth solutions

Let us consider the following problem, elliptic for $\varepsilon > 0$:

$$Q_\varepsilon(u, v) = (f, v) \quad \text{for all} \quad v \in \overline{B}. \tag{12}$$

Here $Q_\varepsilon(u, v) = Q(u, v) + \varepsilon \int_\Omega b^{ij} \partial_i u \cdot \partial_j v + u \cdot v \, dx$, and we assume that the b^{ij} are of class $C^2(\overline{\Omega})$ and that

$$b^{ij}_{kl} = b^{ij}_{lk} = b^{ji}_{kl} \qquad (1 \leq i, j \leq n, 1 \leq k, l \leq p), \tag{13}$$

$$b^{ij}_{kl} z^k_i z^l_j \geq 0, \ (a^{ij}_{kl} + \varepsilon b^{ij}_{kl}) z^k_i z^l_j \geq K_0 \varepsilon |u|^2 \quad \text{in } \Omega \text{ for } z^k_i \in R, \tag{14}$$

K_0 being a positive constant, $|z|^2 = \sum_{i,k} (z^k_i)^2$. We remark that $b^{ij} = \delta^{ij} I$ ($I: p \times p$-unit matrix) has the required properties. Using the conditions (3) and (U) it is easily seen that

$$Q_\varepsilon(u, u) \geq K\varepsilon \| u \|_1^2 + q \| u \|_0^2 \quad \text{for} \quad u \in B, \tag{15}$$

$\| u \|_m$ denoting the norm in $H_m(\Omega)$ for a nonnegative integer m, $K = \min(1, K_0)$. Thus $Q_\varepsilon(u, v)$ is coercive over $\overline{B}$ for $\varepsilon > 0$.

To obtain smooth solutions of (9) we proceed in a way similar to that of Kohn and Nirenberg in [3], [4]: Solve (12) for $\varepsilon > 0$ by $u_\varepsilon \in \overline{B} \cap C^\infty(\overline{\Omega})$, derive a-priori estimates for the H_m-norm of u_ε independent of ε, and conclude that as $\varepsilon \to 0$ a subsequence of the u_ε converges in $H_m(\Omega)$ to a solution of (9).

Let f belong to $C^\infty(\overline{\Omega})$. Then according to [3], p. 463 the unique solution $u_\varepsilon \in \overline{B}$ ($\varepsilon > 0$) of (12) is of class $C^\infty(\overline{\Omega})$, provided that

$$\varphi \in C^\infty(R^n) \ ; \ a^{ij}, b^{ij}, b^i, c \in C^\infty(\overline{\Omega}) \ ; \ A^j, B, Q_+ \in C^\infty(\partial\Omega), \tag{16}$$

(17) Q_+ is of constant rank on each component of $\partial\Omega$ [2].

(2) (17) is assumed to satisfy the conditions (a), (b), (c) of [3], pp. 451, 452.

Now we give a-priori estimates corresponding to those of [4], section 4, which are derived for a single equation and zero boundary datas. It may be verified by a close examination of the proofs used there that by almost the same arguments (see also [3], section 4 and [4], p. 799) it follows in our case of systems and correctly posed boundary conditions:

Assume that (16), (17) [(2)] are satisfied. Suppose that the principal part of $Q_\varepsilon(u, v)$ is symmetric, i.e. $\int_\Omega (a^{ij} + \varepsilon b^{ij})\, \partial_i u \cdot \partial_j v\, dx = \int_\Omega (a^{ij} + \varepsilon b^{ij})\, \partial_i v \cdot \partial_j u\, dx$ (this will be the case if $A^j = 0$ on $\partial\Omega$) and that $\partial\Omega$ is noncharacteristic for L, i.e. $\det(a^{ij} n_i n_j) > 0$ on $\partial\Omega$. If u_ε is the $C^\infty(\overline{\Omega})$-solution of (12) for $\varepsilon \geq 0$, then there exist constants $k > 0$, C_m, K independent of ε such that

$$q \| u_\varepsilon \|_m^2 + k\varepsilon \| u_\varepsilon \|_{m+1}^2 \leq C_m \| u_\varepsilon \|_m^2 + K \| f \|_m^2 . \tag{18}$$

The only interesting constant C_m arises from a considerable number of integrations by parts performed locally tangent to the boundary, after a change of the dependent variables. That is why only the following will be said: C_m depends on m, n, p, $1/\min_{\partial\Omega}(\det(a^{ij} n_i n_j))$, $\max_\Omega |D^\gamma a^{ij}_{kl}|$ $(|\gamma| \leq 2)$, $\max_\Omega |D^\gamma b^i_{kl}|$, $\max |D^\gamma B^i_{kl}|$ $(|\gamma| \leq 1)$, further (because of the change of the independent variables) on $\max_\Omega |D^\gamma \varphi|$ $(|\gamma| \leq 3)$ and (because of the change of the dependent variables) on $\max_{\partial\Omega} |D^\gamma Q_+|$ $(|\gamma| \leq 2)$, the dependence being such that C_m increases if the mentioned quantities increase. Thus, if we assume $C_m > q$ for certain $m \geq 1$ we get the following *a*-priori estimate

$$\| u_\varepsilon \|_m \leq K' \| f \|_m . \tag{19}$$

It follows immediately from Lemma 1 that our smooth solutions u_ε $(\varepsilon > 0)$ constructed above are of class $B^\infty = \{u \in C^\infty(\overline{\Omega}) : Q'_+ u = 0 \text{ on } \partial\Omega\}$. Now from (19) it is concluded in the same way as in [3], [4] that a subsequence of the u_ε converges in $H_m(\Omega)$ to a solution u of (9), u belonging to the closure of B^∞ in $H_m(\Omega)$ which will be denoted by $\overline{B^\infty}^m$. We summarize in

THEOREM 1. *Assume that* (16), (17) *hold. Suppose that* $A^j = 0$ *on* $\partial\Omega$ $(j = 1, \cdots, n)$, *that* $\partial\Omega$ *is nowhere characteristic for* L *and that* $C_m < q$ *for certain* $m \geq 1$. *Then for any vector* $f \in H_m(\Omega)$ *there exists a solution* $u \in \overline{B^\infty}^m$ *of the boundary value problem* (9).

Suppose that $\overline{B^\infty}^m \subset C^2(\overline{\Omega})$. Then the solution u of Theorem 1 satisfies our original boundary value problem (1), (4) in the classical sense. This follows with the aid of Lemma 1 and Lemma 2, observing that (17) implies $\Gamma = \Lambda = \partial\Omega$.

5. L_2–WEAK AND L_2–STRONG SOLUTIONS

DEFINITION 4. *Let* $f \in L_2(\Omega)$. *Then a function* $u \in L_2(\Omega)$ *will be called a* L_2*-weak solution of* (1), (4), *if* $(L^* v, u) = (v, f)$ *for all* $v \in C^2(\overline{\Omega})$ *satisfying* (6),

It was proved in [1] that a correctly posed boundary value problem (1), (4) always has a L_2-weak solution u with the property

$$\inf_{w \in N(L)} \| u + w \| \leq q^{-1} \| f \| , \tag{19}$$

$N(L) = \{u \in L_2(\Omega) : (L^* v, u) = 0 \text{ for } v \in C^2(\overline{\Omega}) \text{ satisfying (6)}\}$. Let us now consider the uniqueness of L_2-weak solutions.

We need some Green's formulae. Corresponding to (7) we get

$$(L^* v, u) = Q^*(v, u) - \int_{\partial\Omega} [Q^j \partial_j v + A^j \partial_j v + C' v] \cdot u \, d\sigma , \tag{20}$$

$Q^*(v, u) = Q(u, v)$. We use the notations $Q^*_\varepsilon(v, u) = Q^*(v, u) + \varepsilon \int_\Omega (b^{ij} \partial_i v \cdot \partial_j u + v \cdot u) \, dx$ and $L^*_\varepsilon = L^* + \varepsilon(- \partial_j b^{ij} \partial_i + I)$. Then we obtain

$$(L^*_\varepsilon v, u) = Q^*_\varepsilon(v, u) - \int_{\partial\Omega} [(Q^j + \varepsilon b^{ij} n_i) \partial_j v + A^j \partial_j v + C' v] \cdot u \, d\sigma. \tag{21}$$

In order to prove uniqueness we first solve smoothly the boundary value problem

$$L^*_\varepsilon v = g \quad \text{in } \Omega , \quad (6) \qquad (\varepsilon > 0) \tag{22}$$

for $g \in C^\infty(\overline{\Omega})$. Assuming (16), (17), the solution $v = v_\varepsilon \in \overline{B}$ of

$$Q^*_\varepsilon(v, u) = (g, u) \quad \text{for } u \in \overline{B} \qquad (\varepsilon > 0)$$

is of class $C^\infty(\overline{\Omega})$. Let us require $b^{ij} n_i = 0$ on $\partial\Omega$, so that the ε-term in the boundary integral of (21) vanishes. Then it follows by Lemma 2, (ii) that v_ε is the desired solution of (22).

Now let $u \in N(L)$ and let v_ε be the $C^\infty(\overline{\Omega})$-solution of (22) corresponding to $g \in C^\infty(\overline{\Omega})$. Then we get $(g, u) = (L^*_\varepsilon v_\varepsilon, u) = \varepsilon((- \partial_j b^{ij} \partial_i + I) v_\varepsilon, u)$, thus

$$|(g, u)| \leq \text{const. } \varepsilon \| v_\varepsilon \|_2 \| u \|_0 .$$

Let us assume that (18) holds with $C_1 < q$. Then

$$|(g, u)| \leq \text{const. } \varepsilon^{1/2} \| g \|_1 \| u \|_0 \quad \text{for } \varepsilon > 0 ;$$

consequently $u = 0$ in $L_2(\Omega)$. We summarize in

THEOREM 2. *Assume the existence of matrices* b^{ij} *satisfying* (13), (14) *and* $b^{ij} n_i = 0$ *on* $\partial\Omega$ $(j = 1, \cdots, n)$. *Let* (16), (17) *hold. Suppose that*

$A^j = 0$ *on* $\partial\Omega$ $(j = 1, \cdots, n)$ *and that* $\partial\Omega$ *is noncharacteristic for* L. *Let in addition* $q > C_1$. *Then* L_2*-weak solutions of* (1), (4) *are unique; the same holds for* L_2*-weak solutions of the formally adjoint problem* (5), (6).

For the case $n = 2$ the following matrices b^{ij} can be used in Theorem 2:

$$b^{11} = (\varphi_{|2}^2 + \varphi)\,I \quad , \quad b^{22} = (\varphi_{|1}^2 + \varphi)\,I \quad , \quad b^{12} = b^{21} = -\varphi_{|1}\,\varphi_{|2}\,I.$$

It is obvious that $b^{ij} n_i = 0$ on $\partial\Omega$ and that (13) and the first condition of (14) are satisfied. The second part of (14) holds because $\det(a^{ij} n_i n_j) > 0$ on $\partial\Omega$.

Finally we use Theorem 2 to show « weak equals strong ».

DEFINITION 5. *Let* $f \in L_2(\Omega)$. *Then a function* $u \in L_2(\Omega)$ *will be called an* L_2*-strong solution of* (1), (4) (*in the sense of K. O. Friedrichs*), *if there exists a sequence* $u_j \in C^2(\overline{\Omega})$ *satisfying* (4) *such that*

$$\| u_j - u \| \to 0 \quad , \quad \| L u_j - f \| \to 0 \qquad (j \to \infty).$$

Clearly any L_2-strong solution is a L_2-weak solution. Conversely we have

COROLLARY. *Suppose that the assumptions of Theorem* 2 *hold. Then a* L_2*-weak solution of* (1), (4) *is actually a* L_2*-strong solution.*

Proof. Using Theorem 2, there is a sequence $u_j \in C^2(\overline{\Omega})$ satisfying (4) such that $\| L u_j - f \| \to 0$ $(j \to \infty)$. The Corollary then follows with the aid of (19), observing that $N(L) = \{0\}$.

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