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# Rendiconti

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## Properties of uniform integrability and convergence for families of random variables

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**Probabilità.** — *Properties of uniform integrability and convergence for families of random variables.* Nota (*) di MAGDA RUBINSTEIN, presentata dal Socio B. SEGRE.

RIASSUNTO. — Sotto opportune condizioni, vien stabilita l'uniforme integrabilità di una famiglia di variabili casuali. Si generalizza inoltre un ben noto risultato sulle sottomartingale.

## 1. INTRODUCTION

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n)_{n\in\mathbf{N}}$ an increasing family of sub $\sigma$-fields of $\mathscr{F}$. In what follows $(X_n)_{n\in\mathbf{N}}$ is a sequence of random variables such that:

$$X_n \quad \text{is} \quad \mathscr{F}_n\text{-measurable}, \qquad \text{and}$$

$$\sum_{K=1}^{\infty} \int |E(Y_{K+1} \mid \mathscr{F}_K)| < \infty \qquad \text{where} \quad Y_{K+1} = X_{K+1} - X_K. \tag{1}$$

Let $\mathscr{T}$ be the set of all stopping times (see [2]) $T : \Omega \to \mathbf{N}$. For $T \in \mathscr{T}$ we denote $\mathscr{F}_T = \{A \subset \Omega \mid A \cap (T = n) \in \mathscr{F}_n\}$. It is known that $\mathscr{F}_T$ is a $\sigma$-field, T is $\mathscr{F}_T$-measurable, and $T_1 \leq T_2$ implies $\mathscr{F}_{T_1} \subset \mathscr{F}_{T_2}$. We denote by $X_T$ the function $\omega \mapsto X_{T(\omega)}^{(\omega)}$ which is $\mathscr{F}_T$-measurable.

In this paper we shall show that from condition (1) and the uniform integrability of $(X_n)_{n\in\mathbf{N}}$ it follows that the family $(X_T)_{T\in\mathscr{T}}$ is uniformly integrable. We shall further show that, under the same conditions, the condition (1) is invariant by passing to an increasing sequence of stopping times, and that the generalized sequence $(X_T)_{T\in\mathscr{T}}$ converges in $L_1$. By Remark 2.1, Theor. 2 is a generalization of a known result on submartingales.

2. We begin with a remark about condition (1).

2.1. REMARK. If $(X_n)_{n\in\mathbf{N}}$ is a submartingale with $\sup_{n\in\mathbf{N}} \int X_n < \infty$, then (1) holds. Indeed, we have

$$E(X_n \mid \mathscr{F}_m) - X_m = \sum_{K=m}^{n-1} E(E(Y_{K+1} \mid \mathscr{F}_K) \mid \mathscr{F}_m), \qquad \text{for} \quad m < n. \tag{2}$$

Thus

$$\int X_n - \int X_m = \sum_{K=m}^{n-1} \int E(Y_{K+1} \mid \mathscr{F}_K) = \sum_{K=m}^{n-1} \int |E(Y_{K+1} \mid \mathscr{F}_K)| \to 0$$

as $n \geq m \to \infty$.

(*) Pervenuta all'Accademia il 2 agosto 1974.

2.2. THEOREM. *If* $(X_n)_{n \in \mathbf{N}}$ *is uniformly integrable and satisfies* (1), *then* $(X_T)_{T \in \mathscr{T}}$ *is uniformly integrable.*

*Proof.* In the hypothesis of the theorem, there exists, according to ([1], Th. 1, p. 81, II), an element $X \in L_1$ such that $X_n \to X$ in $L_1$ and a.e. For $T \in \mathscr{T}$ we have:

$$X_T = X_T - E(X \mid \mathscr{F}_T) + E(X \mid \mathscr{F}_T).$$

In view of ([2]) Th. 19, pp. 119, V) the family $(E(X \mid \mathscr{F}_T))_{T \in \mathscr{T}}$ is uniformly integrable. It remains to prove that $(Z_T)_{T \in \mathscr{T}}$ where $Z_T = X_T - E(X \mid \mathscr{F}_T)$ is uniformly integrable. Since $X_n \to X$ in $L_1$ it follows that

$$E(X_n \mid \mathscr{F}_m) \to E(X \mid \mathscr{F}_m) \quad \text{as} \quad n \to \infty \quad \text{in } L_1, \qquad \text{for every} \quad m \in \mathbf{N}. \tag{3}$$

Let $A \in \mathscr{F}_m$. By (2) we have:

$$\int_A |E(X_n \mid \mathscr{F}_m) - X_m| \leq \int_A \sum_{K=m}^{n-1} |E(Y_{K+1} \mid \mathscr{F}_K)|.$$

Therefore, for $n \to \infty$, by (3) it follows:

$$\int_A |E(X \mid \mathscr{F}_m) - X_m| \leq \int_A \sum_{K=m}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)|. \tag{4}$$

Let $T \in \mathscr{T}$ and $A \in \mathscr{F}_T$. From (4) and the fact that $E(X \mid \mathscr{F}_T) = E(X \mid \mathscr{F}_i)$ a.e. on $\{T = i\}$, we have

$$\int_A |X_T - E(X \mid \mathscr{F}_T)| = \sum_i \int_{\{T=i\} \cap A} |E(X \mid \mathscr{F}_i) - X_i| \leq \sum_i \int_{\{T=i\} \cap A} \sum_{K=i}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)| = \tag{5}$$

$$= \int_A \sum_{K=T}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)| \leq \int_A \sum_{K=1}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)|.$$

Now, by Markov's inequality,

$$P(\{|Z_T| > a\}) \leq \frac{\int |Z_T|}{a} \leq \frac{\sum_{K=1}^{\infty} \int |E(Y_{K+1} \mid \mathscr{F}_K)|}{a}.$$

Therefore $\lim_{a \to \infty} P(|Z_T| > a) = 0$ uniformly with $T \in \mathscr{T}$.

Taking in (5) $A = \{|Z_T| > a\}$ we obtain

$$\int_{\{|Z_T| > a\}} |Z_T| \leq \int_{\{|Z_T| > a\}} \sum_{K=1}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)| \to 0 \qquad \text{as} \quad a \to \infty$$

uniformly with $T \in \mathscr{T}$.

It follows that $(Z_T)_{T \in \mathscr{T}}$ is uniformly integrable, Q.E.D.

2.3. COROLLARY. *If* $(T_n)_{n \in \mathbf{N}}$ *is an increasing sequence of stopping times such that* $T_n \to \infty$ *a.e. as* $n \to \infty$ *and* $(X_n)_{n \in \mathbf{N}}$ *satisfies the hypothesis of the Theor. 2.2, then* $(X_{T_n})_{n \in \mathbf{N}}$ *converges in* $L_1$ *and a.e. to the limit of* $(X_n)_{n \in \mathbf{N}}$.

*Proof.* Let $A = \{\omega \in \Omega \mid T_n(\omega) \not\to \infty\} \cup \{\omega \in \Omega \mid X_n(\omega) \not\to X(\omega)\}$. For $\omega \notin A$, $(X_{T_n(\omega)}^{(\omega)})_{n \in \mathbf{N}}$ is a subsequence of $(X_n(\omega))_{n \in \mathbf{N}}$ which converges to $X(\omega)$. Therefore $X_{T_n} \to X$ a.e. (since $P(A) = 0$). By Theor. 2.2 $(X_{T_n})_{n \in \mathbf{N}}$ is uniformly integrable. It follows that $X_{T_n} \xrightarrow{n} X$ in $L_1$, Q.E.D.

The following corollary is an obvious consequence of Remark 2.1 and Theorem 2.2.

2.4. COROLLARY ([2], Th. 29, p. 216, V). *If* $(X_n)_{n \in \mathbf{N}}$ *is an uniformly integrable submartingale, then the family* $(X_T)_{T \in \mathscr{T}}$ *is uniformly integrable.*

It is known that $\mathscr{T}$ is a directed set.

2.5. PROPOSITION. *In the hypotheses of Theor. 2.2, for* $(X_n)_{n \in \mathbf{N}}$ *the generalized sequence* $(X_T)_{T \in \mathscr{T}}$ *converges in* $L_1$ *to* X, *the* $L_1$ *limit of* $X_n$.

*Proof.* From relation (5) we have

$$\int_A |(E(X \mid \mathscr{F}_T) - X_T)| \leq \int_A \sum_{K=T}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)|, \qquad A \in \mathscr{F}_T.$$

It follows that

$$\int |E(X \mid \mathscr{F}_T) - X_T)| \leq \int \sum_{K=T}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)|. \tag{6}$$

Let $\varepsilon > 0$. From (1) it follows the existence of $N_\varepsilon \in \mathbf{N}$ such that

$$\int \sum_{K=N_\varepsilon}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)| < \varepsilon. \tag{7}$$

If we denote $T_\varepsilon = N_\varepsilon$ then by (6) and (7), we have for $T > T_\varepsilon$

$$\int |E(X \mid \mathscr{F}_T) - X_T| \leq \int \sum_{K=T}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)| \leq \int \sum_{K=T_\varepsilon}^{\infty} |E(Y_{K+1} \mid \mathscr{F}_K)| < \varepsilon.$$

Hence $E(X \mid \mathscr{F}_T) - X_T \xrightarrow[T \in \mathscr{T}]{} 0$ in $L_1$. Since

$$|X_T - X| \leq |E(X \mid \mathscr{F}_T) - X| + |E(X \mid \mathscr{F}_T) - X_T|,$$

it remains to be proved that the generalized sequence $(E(X \mid \mathscr{F}_T) - X)_{T \in \mathscr{T}}$ converges to zero in $L_1$.

It is known that $E(X \mid \mathscr{F}_T)$ converges in $L_1$ to X. It follows that there exists $N_\varepsilon \in \mathbf{N}$ such that

$$\int |E(X \mid \mathscr{F}_{N_\varepsilon}) - X| < \varepsilon/2.$$

Let $T \in \mathcal{T}$, $T > N_\varepsilon$. We have

$$\int |E(X \mid \mathcal{F}_T) - X| = \sum_{i=1}^{\infty} \int_{\{T=N_\varepsilon+i\}} |E(X \mid \mathcal{F}_{N_\varepsilon+i}) - X|. \tag{8}$$

It is easy to see that for, $n < m$ and $A \in \mathcal{F}_m$, we have

$$\int_A |E(X \mid \mathcal{F}_n) - E(X \mid \mathcal{F}_m)| \leq \int_A |E(X \mid \mathcal{F}_n) - X|.$$

It follows that

$$\int_{\{T=N_\varepsilon+i\}} |E(X \mid \mathcal{F}_{N_\varepsilon+i}) - X| \leq \int_{\{T=N_\varepsilon+i\}} |E(X \mid \mathcal{F}_{N_\varepsilon}) - X| + \tag{9}$$

$$+ \int_{\{T=N_\varepsilon+i\}} |E(X \mid \mathcal{F}_{N_\varepsilon}) - E(X \mid \mathcal{F}_{N_\varepsilon+i})| \leq 2 \int_{\{T=N_\varepsilon+i\}} |E(X \mid \mathcal{F}_{N_\varepsilon}) - X|.$$

By (8) and (9), we have:

$$\int |E(X \mid \mathcal{F}_T) - X| \leq 2 \int |E(X \mid \mathcal{F}_{N_\varepsilon}) - X < \varepsilon \quad \text{for } T > N_\varepsilon, \qquad \text{Q.E.D.}$$

2.6. PROPOSITION. *Let* $(T_n)_{n \in \mathbf{N}}$ *be an increasing sequence of stopping times and* $(X_n)_{n \in \mathbf{N}}$ *as in Theor. 2.2. Then* $(X_{T_n})_{n \in \mathbf{N}}$ *satisfies* (1).

*Proof.* Let $A \in \mathcal{F}_{T_n}$. If we put $A_i = A \cap \{T_n = i\}$ then we have

$$\int_A (X_{T_{n+1}} - X_{T_n}) = \sum_i \int_{A_i} (X_{T_{n+1}} - X_i).$$

Further we have:

$$\left| \int_{A_i} (X_{T_{n+1}} - X_i) \right| \leq \left| \sum_{k \geq 0} \int_{A_i \cap \{T_{n+1} > i+k\}} Y_{i+k+1} \right| \leq$$

$$\leq \sum_{j \geq i} \int_{A_i} X_{\{T_{n+1} > j\}} |E(Y_{j+1} \mid \mathcal{F}_j)| = \int_{A_i} \sum_{j=i}^{T_{n+1}-1} |E(Y_{j+1} \mid \mathcal{F}_j)|.$$

It follows

$$\left| \int_A E[(X_{T_{n+1}} - X_{T_n}) \mid \mathcal{F}_{T_n}] \right| \leq \int_A \sum_{j=T_n}^{T_{n+1}-1} |E(Y_{j+1} \mid \mathcal{F}_j)|;$$

hence

$$\int |\mathrm{E}\,[(\mathrm{X}_{\mathrm{T}_{n+1}} - \mathrm{X}_{\mathrm{T}_n}) \mid \mathscr{F}_{\mathrm{T}_n}]| \leq \int \sum_{j=\mathrm{T}_n}^{\mathrm{T}_{n+1}-1} |\mathrm{E}\,(\mathrm{Y}_{j+1} \mid \mathscr{F}_j)|$$

and therefore

$$\sum_{n=1}^{\infty} \int |\mathrm{E}\,[(\mathrm{X}_{\mathrm{T}_{n+1}} - \mathrm{X}_{\mathrm{T}_n}) \mid \mathscr{F}_{\mathrm{T}_n}]| \leq \int \sum_{j=1}^{\infty} |\mathrm{E}\,(\mathrm{Y}_{j+1} \mid \mathscr{F}_j)|, \qquad \text{Q.E.D.}$$

## BIBLIOGRAPHY

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