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Special curvature collineation and projective symmetry in Finsler space

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Geometria differenziale. — *Special curvature collineation and projective symmetry in Finsler space.* Nota di H. D. PANDE e A. KUMAR, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Studio di relazioni fra le connessioni (di Berwald, proiettive) in uno spazio di Finsler, trasformazioni puntuali infinitesime (determinate da un campo di vettori), collineazioni speciali sia di curvatura sia di Ricci secondo le denominazioni di Prasad.

Let us consider an n-dimensional Finsler space F_n [1] (1) in which the metric tensors g_{ij} and $g^{ij}(x, \dot{x})$ are symmetric in their indices i and j and are homogeneous of degree zero in $\dot{x}^i$. The covariant derivative in the sense of Berwald of any vector field $X^i(x, \dot{x})$ is given by

$$X^i_{(k)} = \partial_k X^i + \dot{\partial}_h X^i G^h_k + X^h G^i_{hk}, \tag{1.1}$$

where $\partial_k = \partial/\partial x^k$ and $\dot{\partial}_k = \partial/\partial \dot{x}^k$

and the function $G^i(x, \dot{x})$ is positively homogeneous of degree two in its directional argument. This function satisfies the following identities:

$$G^i_{hk\gamma}\dot{x}^\gamma = 0 \quad , \quad G^i_{hk}\dot{x}^k = G^i_h \quad , \quad \dot{\partial}_h \dot{\partial}_k G^i = G^i_{hk} . \tag{1.2}$$

The projective covariant derivative [2] of a vector field $X^i(x, \dot{x})$ is given by

$$X^i_{((k))} = \partial_k X^i - \dot{\partial}_h X^i \Pi^h_{\gamma k}\dot{x}^\gamma + X^h \Pi^i_{hk}, \tag{1.3}$$

where $\Pi^i_{hk}(x, \dot{x})$ is the projective connection coefficient defined by

$$\Pi^i_{hk}(x, \dot{x}) \overset{\text{def}}{=} G^i_{hk} - \frac{1}{(n+1)}(2\,\delta^i_{(h}G^\gamma_{k)\gamma} + \dot{x}^i G^\gamma_{\gamma kh})^{(2)}. \tag{1.4}$$

We have the following commutation formula:

$$2T^i_{[(s)(m)]} = T^h_j H^i_{smh]} - T^i_h H^h_{smj} - \dot{\partial}_h T^i_j H^h_{sm}, \tag{1.5}$$

where $H^i_{hk}(x, \dot{x})$ is Berwald's curvature tensor which is positively homogeneous of degree one in $\dot{x}^i$. The deviation tensor field $H^i_j(x, \dot{x})$ satisfies the following identities:

$$H^i_{jkh}\dot{x}^j = H^i_{kh} \quad , \quad H^i_i = (n-1)H \quad , \quad H^i_{jk}\dot{x}^j\dot{x}^k = 0 \quad , \quad \dot{\partial}_h H^i_{jk} = H^i_{hjk} . \tag{1.6}$$

(*) Nella seduta del 29 giugno 1974.

(1) The numbers in the square brackets refer to the references at the end of the paper.

(2) $2A_{(hk)} = A_{hk} + A_{kh}$ and $2A_{[hk]} = A_{hk} - A_{kh}$.

Now let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \tag{1.7}$$

where $v^i(x)$ is any vector field and dt an infinitesimal constant. The Lie-derivatives of $T^i_j(x, \dot{x})$ and $G^i_{hk}(x, \dot{x})$ [6] are given by

$$\mathcal{L}T^i_j = T^i_{j(h)}v^h + T^i_h v^h_{(j)} - T^h_j v^i_{(h)} + \dot{\partial}_h T^i_j v^h_{(s)} \dot{x}^s \tag{1.8}$$

and

$$\mathcal{L}G^i_{hk} = v^i_{(h)(k)} + v^\gamma H^i_{\gamma hk} + G^i_{hk\gamma} v^\gamma_{(s)} \dot{x}^s \tag{1.9}$$

respectively. The following commutation formulae have been obtained:

$$\mathcal{L}(\dot{\partial}_k T^i_j) - \dot{\partial}_k(\mathcal{L}T^i_j) = 0\,, \tag{1.10}$$

$$\mathcal{L}(T^i_{j(k)}) - (\mathcal{L}T^i_j)_{(k)} = T^h_j \mathcal{L}G^i_{kh} - T^i_h \mathcal{L}G^h_{kj} - (\dot{\partial}_h T^i_j)\mathcal{L}G^h_{ks}\dot{x}^s \tag{1.11}$$

and

$$(\mathcal{L}G^i_{jh})_{(k)} - (\mathcal{L}G^i_{kh})_{(j)} = \mathcal{L}H^i_{hjk} + (\mathcal{L}G^\gamma_{kl})\,G^i_{\gamma jh}\dot{x}^l - (\mathcal{L}G^\gamma_{jl})\,\dot{x}^l G^i_{\gamma hk}\,. \tag{1.12}$$

The following definitions will be used in later discussions of the paper.

Special Projective motion. Sinha [4] has defined that a F_n is said to admit a projective motion provided there exists a vector v^i such that

$$\mathcal{L}G^i_{jk} = -2\delta^i_{(j}P_{k)} - P_{jk}\dot{x}^i \quad , \quad P_k = \dot{\partial}_k P \quad \text{and} \quad P_{hk} = \dot{\partial}_h \dot{\partial}_k P\,, \tag{1.13}$$

where $P(x, \dot{x})$ is a scalar function, positively homogeneous of degree one in $\dot{x}^i$. We consider a special projective change [1] as defined by

$$P = \frac{1}{(n+1)} G^\gamma_{\gamma s}\dot{x}^s \quad \text{and} \quad P_{hk} = \frac{1}{n+1} G^\gamma_{\gamma kh}\,. \tag{1.14}$$

With the help of equation (1.14) we can write (1.13) in the form [7]

$$\mathcal{L}G^i_{hk} = \Pi^i_{hk} - G^i_{hk}\,. \tag{1.15}$$

Special Curvature Collineation (Prasad [3]). A F_n is said to admit a special curvature collineation if there exists a vector v^i such that

$$\mathcal{L}H^i_{jkh} = 0\,. \tag{1.16}$$

Special Ricci Collineation (Prasad [3]). A F_n is said to admit a special Ricci collineation provided there exists a vector v^i such that

$$\mathcal{L}H_{ij} = 0\,. \tag{1.17}$$

2. SPECIAL CURVATURE COLLINEATION

We shall now consider the condition under which a special projective motion is a special curvature collineation. The Lie-derivative of $H^i_{jkh}(x, \dot{x})$ takes the following form in view of the equation (1.15).

$$(2.1) \qquad \mathcal{L}H^i_{jkh} = 2\{\Pi^i_{j[k(h)]} - G^i_{j[k(h)]} + (G^i_{mj[h}\Pi^m_{k]\gamma} + G^i_{mj[k}G^m_{h]\gamma})\dot{x}^\gamma\}.$$

If a special projective motion for F_n is a special curvature collineation then we get from (1.16) and (2.1).

$$(2.2) \qquad \Pi^i_{[k(h)]} - G^i_{[k(h)]} = 0$$

where we have made use of (1.2) and of the fact that $\dot{x}^i_{(k)} = 0$. Thus we have

THEOREM 2.1. *In a Finsler space* F_n *if a special projective motion becomes a special curvature collineation then equation* (2.2) *holds.*

Contracting (2.1) with respect to indices i and h and using (1.6) we get

$$(2.3) \qquad \mathcal{L}H_{jk} = 2\{\Pi^i_{j[k(i)]} - G^i_{j[k(i)]} + (G^i_{mj[i}\Pi^m_{k]\gamma} - G^i_{mj[i}G^m_{k]\gamma})\dot{x}^\gamma\}.$$

From equations (1.17) and (2.3), we obtain

$$(2.4) \qquad \Pi^i_{j[k(i)]} - G^i_{j[k(i)]} + (G^i_{mj[i}\Pi^m_{k]\gamma} - G^i_{mj[k}G^m_{i]\gamma})\dot{x}^\gamma = 0.$$

Multiplying (2.4) by $\dot{x}^j$ and noting (1.2), we get

$$(2.5) \qquad \Pi^i_{[k(i)]} - G^i_{[k(i)]} = 0.$$

Thus we have

THEOREM 2.2. *A necessary condition for a special projective motion to be a special Ricci collineation in a* F_n *is that* (2.5) *holds.*

3. SPECIAL PROJECTIVE SYMMETRIC FINSLER SPACE

DEFINITION 3.1. *A Finsler space* F_n *is said to be a special projective symmetric space if* (1.15) *holds and its Berwald's curvature tensor satisfies the relation*

$$(3.1) \qquad H^i_{jkh(\gamma)} = 0.$$

In this space the following identities are also satisfied

$$(3.2) \qquad a)\ H^i_{jk(\gamma)} = 0, \qquad b)\ H^i_{j(\gamma)} = 0 \qquad \text{and} \quad c)\ H_{(\gamma)} = 0.$$

By applying the commutation formula (1.11) to the deviation tensor field $H^i_j(x, \dot{x})$ and using (1.15) and (3.2), we get

$$(3.3) \qquad (\mathcal{L}H^i_j)_{(k)} = H^i_h(\Pi^h_{kj} - G^h_{kj}) - H^h_j(\Pi^i_{kh} - G^i_{kh}) + \dot{\partial}_h H^i_j(\Pi^h_{ks} - G^h_{ks})\dot{x}^s.$$

With the help of equations (1.1), (1.3) and (3.3), we obtain

$$(3.4) \qquad (\mathcal{L}H^i_j)_{(k)} = H^i_{j(k)} - H^i_{j((k))} .$$

Thus we have following theorems:

THEOREM 3.1. *In a symmetric Finsler space if an infinitesimal transformation* (1.7) *defines a special projective motion then equation* (3.4) *holds.*

THEOREM 3.2. *If in a symmetric Finsler space an infinitesimal transformation* (1.7) *defines a special projective motion such that the Berwald's and projective covariant derivatives of* $H^i_j(x, \dot{x})$ *with respect to* x^k *becomes equal then* $(\mathcal{L}H^i_j)_{(k)} = 0$.

With the help of equations (1.1), (1.2), (1.6), (1.12) and (1.14), we get

$$(3.5) \qquad \mathcal{L}H^i_{jk} = 2 \left\{ \Pi^i_{[j|\overline{m}|(k)]} - G^i_{m[j(k)]} - \dot{\partial}_n \Pi^i_{[j|\overline{m}|} G^n_{k]} \right\} \dot{x}^m \text{ (3)}.$$

From equation (1.1) and operator $\dot{\partial}_k$ and (k) we obtain the following commutation formula

$$(3.6) \qquad \dot{\partial}_h (H^i_{jk(m)}) - (\dot{\partial}_h H^i_{jk})_{(m)} = H^s_{jk} G^i_{hms} - 2 H^i_{s[k} G^s_{j]hm} ,$$

which reduces in view of (3.1) to

$$(3.7) \qquad H^s_{jk} G^i_{hms} + H^i_{sj} G^s_{hkm} - H^i_{sk} G^s_{jhm} = 0 .$$

Taking the Lie-derivative of (3.7) and by using equations (1.15) and (3.5), we get

$$(3.8) \qquad H^s_{jk} (\Pi^i_{hms} - G^i_{hms}) + 2 G^i_{hms} (\Pi^s_{[j|\overline{\gamma}|(k)]} - G^s_{\gamma[j(k)]} + \dot{\partial}_p \Pi^s_{[j|\overline{\gamma}|} G^p_{k]}) \dot{x}^\gamma +$$
$$+ 2 H^i_{s[j} (\Pi^s_{k]hm} - G^s_{k]hm}) + 2 G^s_{khm} (\Pi^i_{[s|\overline{\gamma}|(j)]} - G^i_{\gamma[s(j)]} + \dot{\partial}_p \Pi^i_{s|\overline{\gamma}|} G^p_{j]}) \dot{x}^\gamma -$$
$$- 2 G^s_{jhm} (\Pi^i_{[s|\overline{\gamma}|(k)]} - G^i_{\gamma[s(k)]} + \dot{\partial}_p \Pi^i_{[s|\overline{\gamma}|} G^p_{k]}) \dot{x}^\gamma = 0 .$$

Transvecting equation (3.8) by $\dot{x}^m$ and by using the homogeneity properties of the functions $G^i(x, \dot{x})$ and $\Pi^i_{jk}(x, \dot{x})$, we obtain

$$(3.9) \qquad H^s_{jk} + 2 H^i_{s[j} \Pi^s_{k]hm} = 0 .$$

Thus we have

THEOREM 3.3. *If in a symmetric Finsler space* F_n *an infinitesimal transformation defines a special projective motion then equation* (3.9) *holds.*

(3) The indices in bracket □ are free from symmetric and skew symmetric parts.

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