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Recurrent Finsler spaces with pseudoprojective tensor field

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Geometria differenziale. — *Recurrent Finsler spaces with pseudo projective tensor field.* Nota di H. D. PANDE e B. SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Si studiano gli spazi di Finsler ricorrenti in relazione alla derivazione covariante di Berwald.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n in which the metric tensors $g_{ij}(x, \dot{x})$ and $g^{ij}(x, \dot{x})$ are symmetric in their indices i and j and are positively homogeneous of degree zero in their directional arguments. The Berwald's covariant derivative of a vector field $X^i(x, \dot{x})$ is defined by

$$X^i_{(h)} = \partial_h X^i - \dot{\partial}_m X^i G^m_h + X^m G^i_{mh}. \tag{1.1}$$

In view of (1.1), we have

$$g_{ij(h)} = -2A_{ijh|\gamma}\, l^\gamma \quad , \quad A_{ihj} \overset{\text{def}}{=} FC_{ihj}. \tag{1.2}$$

The H–recurrent Finsler space [2] [1] is characterized by

$$H^i_{jkh(l)} = \lambda_l H^i_{jkh} \tag{1.3}$$

where $\lambda_l(x, \dot{x})$ is a non-null vector field.

The pseudo-projective deviation tensor field $W^{*i}_j(x, \dot{x})$, [5], is defined by

$$W^{*i}_j = a(x, \dot{x})\, W^i_j + b(x, \dot{x})\, H^i_j \tag{1.4}$$

where W^i_j and H^i_j are the projective deviation and Berwald's deviation tensor fields respectively. The functions a and b are homogeneous of degree zero in their directional argument. The pseudo-projective tensor field $W^{*i}_{lhj}(x, \dot{x})$ is defined by

$$W^{*i}_{lhj}(x, \dot{x}) = \dot{\partial}_l W^{*i}_{hj} = \frac{2}{3}\, \dot{\partial}^2_{l[h} W^{*i}_{j]}, \tag{1.5}$$

where

$$W^{*i}_{lhj} = a W^i_{lhj} + b H^i_{lhj} + \dot{\partial}_l a W^i_{hj} + \dot{\partial}_l b H^i_{hj} + \frac{2}{3}\{\dot{\partial}^2_{l[h} a W^i_{j]} + \dot{\partial}_{[h} a\, \dot{\partial}_{(l)} W^i_{j]} + \dot{\partial}^2_{l[h} b H^i_{j]} + \dot{\partial}_{[h} b \dot{\partial}_{(l)} H^i_{j]}\}. \tag{1.6}$$

(*) Nella seduta del 29 giugno 1974.

(1) Numbers in brackets refer to the references at the end of the paper.

In virtue of the homogeneity property of $W^{*i}_{\ j}(x, \dot{x})$, we have the following identities and contractions:

$$(1.7)\qquad (a)\quad W^{*i}_{lhj}\,\dot{x}^l = W^{*i}_{hj} \qquad (b)\quad W^{*i}_{hj}\,\dot{x}^h = W^{*i}_{\ j},$$

$$(1.8)\qquad W^{*i}_{\ i} = b\,(n-1)\,H.$$

2. Recurrent Finsler spaces in the sense of Berwald's covariant derivative

DEFINITION 2.1. An n-dimensional Finsler space F_n is said to be W^*-recurrent if the tensor field $W^{*i}_{lhj}(x, \dot{x})$ satisfies the relation

$$(2.1)\qquad W^{*i}_{lhj(k)} = \lambda_k\, W^{*i}_{lhj}.$$

Transvecting (2.1) with $\dot{x}^l$ and $\dot{x}^h$ successively, we get

$$(2.2)\qquad W^{*i}_{hj(k)} = \lambda_k\, W^{*i}_{hj}$$

and

$$(2.3)\qquad W^{*i}_{\ j(k)} = \lambda_k\, W^{*i}_{\ j}.$$

Hence from (2.2) and (2.3) it is clear that the tensor fields $W^{*i}_{hj}(x, \dot{x})$ and $W^{*i}_{\ j}(x, \dot{x})$ are also recurrent in a W^*-recurrent Finsler space and they are called W^{*i}_{hj}-recurrent and $W^{*i}_{\ j}$-recurrent F_n.

THEOREM 2.1. *If a W^{*i}_{hj}-recurrent Finsler space is projectively flat and H-recurrent, then the following relation holds*:

$$(2.4)\qquad \dot{x}^k\,\{a\,W^i_{lhj(k)} + b_{(k)}\,H^i_{lhj} + N^i_{lhj(k)} - \lambda_k\,N^i_{lhj}\} = \dot{x}^k\,(\dot{\partial}_l\,\lambda_k)\,W^{*i}_{hj},$$

where

$$(2.5)\qquad N^i_{lhj}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_l\,a\,W^i_{hj} + \dot{\partial}_l\,b\,H^i_{hj} + $$
$$+ \frac{2}{3}\,\{\dot{\partial}^2_{l[h}\,a\,W^i_{j]} + \dot{\partial}_{[h}\,a\,\dot{\partial}_{(l)}\,W^i_{j]} + \dot{\partial}^2_{l[h}\,b\,H^i_{j]} + \dot{\partial}_{[h}\,b\,\dot{\partial}_{(t)}\,H^i_{j]}\}.$$

Proof. With the help of the commutation formula $(\dot{\partial}_k\,X^i)_{(h)} = \dot{\partial}_k\,X^i_{(h)} - X^\gamma\,G^i_{\gamma kh}$, we get for the tensor $W^{*i}_{hj}(x, \dot{x})$

$$(2.6)\qquad (\dot{\partial}_l\,W^{*i}_{hj})_{(k)} - \dot{\partial}_l\,W^{*i}_{hj(k)} = -W^{*\gamma}_{hj}\,G^i_{\gamma lk} + W^{*i}_{\gamma j}\,G^\gamma_{hlk} + W^{*i}_{h\gamma}\,G^\gamma_{jlk}.$$

Differentiating (2.2) partially with respect to $\dot{x}^l$ and using the equation (2.6), we obtain

$$(2.7)\qquad (\dot{\partial}_l\,W^{*i}_{hj})_{(k)} - \dot{\partial}_l\,\lambda_k\,W^{*i}_{hj} - \lambda_k\,\dot{\partial}_l\,W^{*i}_{hj} = -W^{*\gamma}_{hj}\,G^i_{lk\gamma} + W^{*i}_{\gamma j}\,G^\gamma_{hlk} + W^{*i}_{h\gamma}\,G^\gamma_{jlk}.$$

Multiplying (2.7) by $\dot{x}^k$ and using the fact that $G^i_{mkh}\dot{x}^k = 0$ and applying the relation (1.5), we have

$$\dot{x}^k \dot{\partial}_l \lambda_k W^{*i}_{hj} = \{ W^{*i}_{lhj(k)} - \lambda_k W^{*i}_{lhj} \} \dot{x}^k .$$

Substituting the value of $W^{*i}_{lhj}(x, \dot{x})$ from (1.6) in the above equation and using (2.5), we obtain

$$\dot{x}^k \dot{\partial}_l \lambda_k W^{*i}_{hj} = \{ a W^i_{lhj} + b H^i_{lhj} + N^i_{lhj} \}_{(k)} - \tag{2.8}$$

$$- \lambda_k \{ a W^i_{lhj} + b H^i_{lhj} + N^i_{lhj} \}$$

Since the space is projectively flat (i.e. $W^i_{lhj} = 0$) and H-recurrent, then (2.8) yields (2.4) in view of the equation (1.3).

THEOREM 2.2. *If a* W^**-recurrent Finsler space is* W^{*i}_j*-recurrent, then we have*

$$g_{ik} (M^i_{lhmj} - M^i_{ljmh}) \dot{x}^l \dot{x}^k = 0 \tag{2.9}$$

where

$$M^i_{lhmj}(x, \dot{x}) \overset{\text{def}}{=} \frac{2}{3} \{ \dot{\partial}^2_{lh} \lambda_m W^{*i}_j + \dot{\partial}_l W^{*i}_j \dot{\partial}_h \lambda_m + \tag{2.10}$$

$$+ \dot{\partial}_l \lambda_m \dot{\partial}_h W^{*i}_j + \dot{\partial}_l W^{*\gamma}_j G^i_{h\gamma m} + \dot{\partial}_l G^i_{\gamma mh} W^{*\gamma}_j \} .$$

Proof. The pseudo-projective tensor field $\overset{*}{W}_{lkhj}(x, \dot{x})$, [5], is given by

$$\overset{*}{W}_{lkhj}(x, \dot{x}) \overset{\text{def}}{=} g_{ik} W^{*i}_{lhj}(x, \dot{x}) . \tag{2.11}$$

Differentiating (2.11) covariantly with respect to x^m and using equations (1.2) and (2.11), we get

$$\overset{*}{W}_{lkhj(m)} - \lambda_m \overset{*}{W}_{lkhj} = g_{ki(m)} W^{*i}_{lhj} = - 2 A_{kim|\gamma} l^\gamma W^{*i}_{lhj} . \tag{2.12}$$

The tensor field $\overset{*}{W}_{lkhj}(x, \dot{x})$ is recurrent in W^*-recurrent F_n if and only if the right hand member of (2.12) vanishes. We obtain

$$(\dot{\partial}^{2\,(2)}_{ph} W^{*i}_j)_{(l)} = \partial_l (\dot{\partial}^2_{ph} W^{*i}_j) - G^\gamma_l \dot{\partial}_\gamma (\dot{\partial}^2_{ph} W^{*i}_j) + \tag{2.13}$$

$$+ (\dot{\partial}^2_{ph} W^{*\gamma}_j) G^i_{\gamma l} - (\dot{\partial}^2_{ph} W^{*i}_\gamma) G^\gamma_{jl} - (\dot{\partial}^2_{\gamma h} W^{*i}_j) G^\gamma_{pl} - (\dot{\partial}^2_{p\gamma} W^{*i}_j) G^\gamma_{hl}$$

and

$$\dot{\partial}^2_{ph} W^{*i}_{j(l)} = \dot{\partial}^2_{ph} (\partial_l W^{*i}_j) - \dot{\partial}^2_{ph} (G^\gamma_l \dot{\partial}_\gamma W^{*i}_j) + \tag{2.14}$$

$$+ \dot{\partial}^2_{ph} (W^{*\gamma}_j G^i_{\gamma l}) - \dot{\partial}^2_{ph} (W^{*i}_\gamma G^\gamma_{jl}) .$$

(2) $\dot{\partial}^2_{ph} \overset{\text{def}}{=} \dot{\partial}_p \dot{\partial}_h$.

From equations (2.13) and (2.14), we get the following commutation formula:

$$(2.15)\qquad (\dot\partial^2_{ph} W^{*i}_{\ j})_{(l)} - \dot\partial^2_{ph} W^{*i}_{\ j(l)} = - G^{\gamma}_{phl}\,\dot\partial_\gamma W^{*i}_{\ j} + \dot\partial_p W^{*\gamma}_{\ j} G^i_{h\gamma l} + \dot\partial_h W^{*\gamma}_{\ j} G^i_{h\gamma l} + \dot\partial_h W^{*\gamma}_{\ j} G^i_{p\gamma l} + W^{*\gamma}_{\ j}\dot\partial_p G^i_{h\gamma l} - \dot\partial_h W^{*i}_{\ \gamma} G^{\gamma}_{pjl} - \dot\partial_p W^{*i}_{\ \gamma} G^{\gamma}_{hjl} - W^{*i}_{\ \gamma}\dot\partial_p G^{\gamma}_{hjl}\,.$$

Differentiating the identity $W^*_{lkhj} + W^*_{kljh} = \frac{2}{3}\{g_{ik}\,\dot\partial^2_{l[h} W^{*i}_{\ j]} + g_{il}\,\dot\partial^2_{k[j} W^{*i}_{\ h]}\}$, [5], covariantly with respect to x^m and using relations (2.11) and (1.5), we obtain

$$(2.16)\qquad \lambda_m (W^*_{lkhj} + W^*_{kljh}) = \frac{2}{3}\{g_{ik}(\dot\partial^2_{l[h} W^{*i}_{\ j]})_{(m)} + g_{il}(\dot\partial^2_{k[j} W^{*i}_{\ h]})_{(m)}\}\,.$$

Substituting the value of $\lambda_m (W^*_{lkhj} + W^*_{kljh})$ in (2.16) and applying the commutation formula (2.15), we get

$$(2.17)\qquad \frac{4}{3}[g_{ik}\{(\dot\partial^2_{l[h} W^{*i}_{\ j](m)} + \dot\partial_l W^{*\gamma}_{[j} G^i_{h]\gamma m} + \dot\partial_l G^i_{\gamma m[h} W^{*\gamma}_{j]} - \lambda_m g_{ik}(\dot\partial^2_{l[h} W^{*i}_{j]})]\,\dot x^l \dot x^k = 0\,.$$

Since W^*-recurrent F_n is also $W^{*i}_{\ j}$-recurrent F_n, then differentiating (2.3) partially with respect to $\dot x^l$ and $\dot x^h$, we get

$$(2.18)\qquad \dot\partial^2_{lh} W^{*i}_{\ j(m)} = \dot\partial^2_{lh}\lambda_m W^{*i}_{\ j} + \dot\partial_h\lambda_m W^{*i}_{\ j} + \dot\partial_l\lambda_m\,\dot\partial_h W^{*i}_{\ j} + \dot\partial^2_{lh} W^{*i}_{\ j}\cdot\lambda_m\,.$$

With the help of equations (2.10), (2.17) and (2.18) we obtain the required result.

THEOREM (2.3). *In an n-dimensional* W^* *recurrent* F_n *the following relation is true*:

$$(2.19)\qquad \dot x^k\{\dot\partial_l\lambda_{k(s)} W^{*i}_{hj} + \dot\partial_h\lambda_{k(s)} W^{*i}_{\ jl} + \dot\partial_j\lambda_{k(s)} W^{*i}_{lh}\} = 0\,.$$

Proof. Differentiating (2.2) partially with respect to $\dot x^l$, we have

$$(2.20)\qquad \dot\partial_l W^{*i}_{\ hj(k)} = \dot\partial_l\lambda_k W^{*i}_{\ hj} + \lambda_k\dot\partial_l W^{*i}_{\ hj}\,.$$

From equations (2.1), (2.6) and (2.20), we obtain

$$(2.21)\qquad (\dot\partial_l\lambda_k) W^{*i}_{hj} = W^{*\gamma}_{hj} G^i_{\gamma lk} - W^{*i}_{\gamma j} G^{\gamma}_{hlk} - W^{*i}_{h\gamma} G^{\gamma}_{jlk}\,.$$

Interchanging the indices l, h and j in (2.21) and adding all the three equations thus obtained, we get

$$(2.22)\qquad (\dot\partial_l\lambda_k) W^{*i}_{hj} + (\dot\partial_h\lambda_k) W^{*i}_{jl} + (\dot\partial_j\lambda_k) W^{*i}_{lh} = W^{*\gamma}_{hj} G^i_{\gamma lk} + W^{*\gamma}_{jl} G^i_{\gamma hk} + W^{*\gamma}_{lh} G^i_{\gamma jk}\,.$$

Differentiating (2.22) covariantly with respect to x^s, and applying the commutation formula (2.6) and using the relation (2.2), we get

$$(2.23) \qquad (\dot{\partial}_l \lambda_{k(s)} + \lambda_\gamma G^\gamma_{lsk}) W^{*i}_{hj} + (\dot{\partial}_h \lambda_{k(s)} + \lambda_\gamma G^\gamma_{hsk}) W^{*i}_{jl} + \\ + (\dot{\partial}_j \lambda_{k(s)} + G^\gamma_{jsk} \lambda_\gamma) W^{*i}_{lh} = W^{*\gamma}_{hj} G^i_{\gamma lk(s)} + W^{*\gamma}_{jl} G^i_{\gamma hk(s)} + W^{*\gamma}_{lh} G^i_{\gamma jk(s)} .$$

Multiplying (2.23) by $\dot{x}^k$ and using the symmetric property of G^i_{hjk}, we get the required result.

References

[1] R. B. MISRA and F. M. MEHER (1972) – *Lie differentiation and projective motion in the projective Finsler space*, «Tensor», N. S., *23*, 57–65.

[2] R. B. MISRA (1973) – *On a recurrent Finsler space*, «Rev. Roum. Math. Pures et appl.», *18*, 701–712.

[3] R. S. MISHRA and H. D. PANDE (1969) – *Recurrent Finsler space*, «J. Ind. Math. Soc.», *32*, 17–22.

[4] H. RUND (1959) – *The differential geometry of Finsler spaces*, Springer Verlag, Berlin.

[5] B. B. SINHA and S. P. SINGH (1971) – *On Pseudo-Projective Tensor Fields*, «Tensor», N. S., *22*, 317–320.