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On α-Lipschitz retractions of the unit closed ball onto its boundary

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Analisi funzionale. — *On α-Lipschitz retractions of the unit closed ball onto its boundary* (*). Nota (**) di MASSIMO FURI e MARIO MARTELLI, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia D il disco unitario di uno spazio di Banach. Si prova che ∂D è un retratto α–Lipschitziano di D se e solo se esiste $k > 0$ ed un'omotopia $H : \partial D \times [0, 1] \to \partial D$ tale che $H(x, 0) = x_0$, $H(x, 1) = x$ e $\alpha(H(A \times [0, t])) \leq tk\alpha(A)$ per ogni $A \subset \partial D$.

1. Let D be the unit closed ball of a Banach space E and $f : D \to D$ be a Lipschitz map with constant $k \geq 1$. In [2] K. Goebel proved that

$$\eta(f) \leq 1 - 1/k$$

where $\eta(f) = \inf\{\| x - f(x)\| : x \in D\}$.

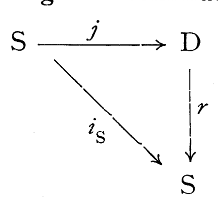
It is not known whether there are Banach spaces such that $\eta(f) = 0$ for any Lipschitz map $f : D \to D$. However K. Goebel pointed out that when E is a Hilbert space there are Lipschitz maps $f : D \to D$ with $\eta(f) > 0$ if and only if S, the boundary of D, is a Lipschitz retract of D.

In [4] we proved that Goebel's inequality holds also for α-Lipschitz maps. Moreover we showed that, in any Banach space, S is an α-Lipschitz retract of D if and only if there exists an α-Lipschitz map $f : D \to D$ with $\eta(f) > 0$.

In this paper we succeeded in constructing an example of a Lipschitz map $f : D \to D$ such that $\eta(f) > 0$. Moreover we gave another formulation of the problem of finding an α-Lipschitz retraction $r : D \to S$, which involves the contractibility of S. This formulation is given in terms of the existence of a particular homotopy $H : S \times [0, 1] \to S$ joining the identity and a constant map.

2. Let E be an infinite dimensional Banach space. In this paper we always denote by S, D the unit sphere and the unit closed ball of E respectively. We will use the following well-known results concerning S and D.

i) S is a retract of D, i.e. there exists a continuous map $r : D \to S$ which makes the following diagram commutative



where j is the inclusion and i_S is the identity on S.

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ii) S is contractible, i.e. there exists a continuous homotopy $H: S \times [0,1] \to S$ joining the identity and a constant map.

Let A be a bounded subset of E. We denote by $\alpha(A)$ the infimum of all $\varepsilon > 0$ such that A can be covered with a finite family of subsets with diameter less than ε (see Kuratowski [6]). We will use the following properties of α.

1) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$, $A, B \subset E$.

2) $\alpha(A) = 0$ if and only if $\bar{A}$ is compact, where $\bar{A}$ is the closure of A.

3) $\alpha(\overline{co}A) = \alpha(A)$ (G. Darbo [1]), where $\overline{co}A$ denotes the closed convex hull of A.

4) $\alpha([0, T] \cdot A) = T\alpha(A)$, where $[0, T] \cdot A = \{tx : 0 \leq t \leq T, x \in A\}$.

Let $M \subset E$ and $f: M \to E$ be a continuous map. If there exists $k \geq 0$ such that $\alpha(f(A)) \leq k\alpha(A)$ for any bounded set $A \subset M$ then f is said to be α-Lipschitz with constant k. In the case when $k < 1$ ($k = 1$) f is called α-contractive (α-nonexpansive). We recall the following result concerning α-contractive maps [1].

Let $f: C \to C$ be an α-contractive map defined on a closed bounded and convex subset of a Banach space E. Then f has a fixed point.

Using the above result it can be easily seen that an α-nonexpansive map $f: C \to C$ is such that $\eta(f) = 0$ [6].

In this paper we deal with α-Lipschitz maps defined in the unit closed ball D of a Banach space E. The fact that $\eta(f) > 0$ will play a key role in our considerations. Therefore we will always assume that the constant k is bigger than 1.

3. Let E_0 be the subspace of $C[0, 1]$ consisting of all functions x such that $x(0) = 0$. The following example shows that $S \subset E_0$ is an α-Lipschitz retract of D.

We point out that the problem of whether S is an α-Lipschitz retract of D in any infinite dimensional Banach space is still open.

Example. Consider the map $\varphi: D \to D$ defined by

$$\varphi(x)(t) = g((1 - t)|x(t)| + t)$$

where

$$g(\tau) = \begin{cases} k\tau, & 0 \leq \tau \leq 1/k \\ 1, & 1/k \leq \tau \leq 1, \end{cases} \qquad k > 1.$$

Since g is continuous and piecewise differentiable with $|g'(\tau)| \leq k$, g is Lipschitz with constant k. It follows that

$$\begin{aligned} |\varphi(x)(t) - \varphi(y)(t)| &= |g((1-t)|x(t)| + t) - g((1-t)|y(t)| + t)| \leq \\ &\leq k|(1-t)|x(t)| + t - (1-t)|y(t)| - t| = k(1-t)||x(t)| - |y(t)|| \leq \\ &\leq k|x(t) - y(t)|. \end{aligned}$$

Thus $\varphi : D \to D$ is a Lipschitz map with constant k. We need only to prove that $\eta(\varphi) > 0$. Let $x \in D$. Clearly there exists $t_0 \in (0, 1)$ such that $(1 - t_0)|x(t_0)| + t_0 = 1/k$. This implies that $\varphi(x)(t_0) = g(1/k) = 1$. On the other hand $|x(t_0)| = (1/k - t_0)/(1 - t_0)$. Therefore

$$\| \varphi(x) - x \| \geq | \varphi(x)(t_0) - x(t_0) | > 1 - 1/k$$

and so $\eta(\varphi) \geq 1 - 1/k$.

The following theorem shows that if S is an α-Lipschitz retract of D then for any $\varepsilon > 0$ there exists an α-Lipschitz map $f : D \to D$ with constant $1 + \varepsilon$ such that $\eta(f) > 0$.

THEOREM 1. *Let Q be a bounded closed and convex subset of a Banach space E. Assume that there exists $k_0 > 1$ such that any α-Lipschitz map $f : Q \to Q$ with constant k_0 has the property $\eta(f) = 0$. Then any α-Lipschitz selfmap of Q has the same property.*

Proof. Let f be an α-Lipschitz selfmap of Q with constant $k > 1$ and let $0 < \lambda < 1$. Define $f_\lambda(x) = (1 - \lambda)x + \lambda f(x)$ for any $x \in Q$. The map f_λ is α-Lipschitz with constant $1 - \lambda + \lambda k$. Moreover $\eta(f_\lambda) = \lambda \eta(f)$. Since there exists $0 < \lambda_0 < 1$ such that $1 - \lambda_0 + \lambda_0 k \leq k_0$ and any α-Lipschitz map g with constant less than k_0 has $\eta(g) = 0$ the theorem is proved. Q.E.D.

It is known that the unit sphere S in an infinite dimensional Banach space is contractible, i.e. there exists a continuous homotopy $H : S \times [0, 1] \to S$ joining the identity and a constant map.

Theorem 2 below shows that the problem of finding an α-Lipschitz retraction $r : D \to S$ is equivalent to the one of the existence of a continuous homotopy $H : S \times [0,1] \to S$ with particular properties. We need the following Proposition.

PROPOSITION 1. *Let X be a complete metric space and let $\mathscr{F}$ be a family of subsets of X such that for any $\varepsilon > 0$ there exists a finite subfamily $\{F_1, F_2, \cdots, F_n\}$ of $\mathscr{F}$ with the property that $\alpha(X / \cup_i F_i) < \varepsilon$. Assume that the restrictions f/F, $F \in \mathscr{F}$, of a continuous map $f : X \to X$ are α-Lipschitz with constant k. Then f is α-Lipschitz with the same constant.*

Proof. Assume first that $\mathscr{F}$ admits a finite subfamily $\{F_i : i = 1, 2, \cdots, n\}$ which is a covering of X. If A is any bounded subset of X we have

$$\alpha(f(A)) = \max\{\alpha(f(A \cap F_i)) : i = 1, 2, \cdots, n\} \leq$$
$$\leq \max\{k\alpha(A \cap F_i) : i = 1, 2, \cdots n\} = k\alpha(A).$$

Assume now that $\mathscr{F}$ has not the above property. Consider the family $\mathscr{B}$ consisting of the sets which are complements of finite unions of elements of $\mathscr{F}$. Clearly $\mathscr{B}$ is a filterbase. Moreover by assumption $\inf\{\alpha(B) : B \in \mathscr{B}\} = 0$. Therefore, by a result proved in [3], we have

a) the set $K = \cap\{\overline{B} : B \in \mathscr{B}\}$ is non-empty and compact.

b) For any neighborhood U of K there exists $B \in \mathscr{B}$ such that $B \subset U$.

Let A be a bounded subset of X. We want to prove that $\alpha(f(A)) \leq k\alpha(A)$. This is true if $\alpha(A) = 0$ since $\bar{A}$ is compact. Assume $\alpha(A) > 0$. There exists a neighborhood $V \supset f(K)$ such that $\alpha(V) < k\alpha(A)$. Let U be a neighborhood of K such that $f(U) \subset V$. There exists a finite subfamily $\{F_1, F_2, \cdots, F_n\}$ of $\mathcal{F}$ with the property that $\cup_i F_i \subset X/U$. Put $A_0 = A \cap U$ and $A_i = A \cap F_i$, $i = 1, 2, \cdots, n$. We have

$$\alpha(f(A)) = \alpha\left\{\bigcup_0^n {}_i\, f(A_i)\right\} = \max\{\alpha(f(A_i)) : i = 0, 1, \cdots, n\} \leq$$

$$\leq \max\{\alpha(V), k\alpha(A)\} = k\alpha(A). \quad \text{Q.E.D.}$$

THEOREM 2. *Let* D *be the unit closed ball in a Banach space* E *and let* S *be the boundary of* D. *The following two conditions are equivalent*

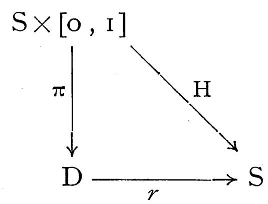
i) *there exists an* α*-Lipschitz retraction* $r: D \to S$ *with constant* k,

ii) *there exists a homotopy* $H: S \times [0, 1] \to S$, *joining the identity and a constant map such that*

$$\alpha(H(A \times [0, t])) \leq tk\alpha(A)$$

for any $A \subset S$ *and* $t \in [0, 1]$.

Proof. i) $\Rightarrow$ ii). Define $H(x, t) = r(tx)$. We have $\alpha(H(A \times [0, t])) = = \alpha(r([0, t] \cdot A)) \leq k\alpha([0, t] \cdot A) = tk\alpha(A)$. ii) $\Rightarrow$ i). The map $\pi: S \times [0, 1] \to D$ defined by $\pi(x, t) = tx$ is a quotient map. Since H is constant in the set $\pi^{-1}(0)$ there exists a unique continuous map r which makes the following diagram commutative



Obviously r is a retraction since $H(x, 1) = x$ for any $x \in S$.

Let $0 < q < 1$ and put $F_n = \{x \in D : q^{n+1} \leq \|x\| \leq q^n\}$, $n = 0, 1, \cdots$ The family $\{F_n : n = 0, 1, \cdots\}$ satisfies the assumption of Proposition 1.

Let us prove now that r/F_n, $n = 0, 1, \cdots$ is α–Lipschitz with constant k/q. Take $A \subset F_n$ and put $\hat{A} = \{x/\|x\| : x \in A\}$. Since $\hat{A} \subset [0, 1/q^{n+1}]$. A we have $\alpha(\hat{A}) \leq \alpha(A)/q^{n+1}$. On the other hand $r(A) \subset H(\hat{A} \times [0, q^n]$. Therefore

$$\alpha(r(A)) \leq q^n\, k\alpha(\hat{A}) \leq k\alpha(A)/q\,.$$

By the above Proposition it follows that r is α-Lipschitz with constant k/q. Since this holds for any $q < 1$ we have that r is α-Lipschitz with constant k. Q.E.D.

REFERENCES

[1] DARBO G. (1955) – *Punti uniti in trasformazioni a codominio non compatto*, « Rend. Sem. Mat. Univ. di Padova », *24*, 353–367.

[2] GOEBEL K. (1973) – *On the minimal displacement of points under lipschitzian mappings*, « Pac. Jour. of Math. », *45*, 151–163.

[3] FURI M. and MARTELLI M. (1970) – *A characterization of compact filterbasis in complete metric spaces*, « Rend. Ist. Mat. Univ. di Trieste », *2*, 109–113.

[4] FURI M. and MARTELLI M. (1974) – *On the minimal displacement of points under α-Lipschitz maps in Normed Spaces*, « Boll. Un. Mat. Ital. », (4), *9*, 791–799.

[5] FURI M. and VIGNOLI A. (1970) – *On α-nonexpansive mappings and fixed points*, « Atti Acc. Naz. Lincei », *48*, 195–198.

[6] KURATOWSKI C. (1958) – *Topologie*, « Monografie Matematiczne », *20*, Warzawa.