Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

J. O. C. Ezeilo

Periodic solutions of certain third order differential equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.1-2, p. 54–60.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_1-2_54_0>

Equazioni differenziali. — *Periodic solutions of certain third order differential equations.* Nota (*) di J. O. C. EZEILO, presentata dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota considera le equazioni della forma

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + \theta(x) = p(t) + q(t, x, \dot{x}) \tag{1}$$

dove $\psi, \varphi, \theta, p, q$ sono funzioni continue dei loro argomenti $p(t+\omega) = p(t)$, $q(t+\omega, x, y) = q(t, x, y)$ per $\omega > 0$, ω costante, e t, x, y qualunque. Nel caso speciale $q \equiv 0$ e $\int_0^t p(s)\,ds$ limitato per t qualunque allora la (1) ha una soluzione ω-periodica se esistono due costanti $a \neq 0$, $h \geq 0$ tali che

i) $x\theta(x) \operatorname{sgn} a \leq 0$, $(|x| \geq h)$; ii) $\left(\int_0^y \psi(s)\,ds - ay\right) = o(1)$ per $|y| \to \infty$.

Se i) è sostituita dalla condizione più restrittiva $x\theta(x) \operatorname{sgn} a \leq -\delta < 0$ $(|n| \geq h)$ l'esistenza di una soluzione periodica vale per l'equazione (1) per $|q(t, x, y)| \leq \alpha + \beta|x|$ con $\alpha + \beta$ costanti e $\beta < \delta$.

1. Consider the differential equation

$$\dddot{x} + a\ddot{x} + \varphi(x)\dot{x} + \theta(x) = p(t) \tag{1.1}$$

where $a \neq 0$ is a constant and $\varphi(x)$, $\theta(x)$ and $p(t)$ are continuous functions depending only on the arguments shown, and $p(t)$ is ω-periodic in t, that is $p(t+\omega) = p(t)$ for some real number $\omega \neq 0$.

In Appendix 3 of his paper [1] Reissig showed that if $P(t) \equiv \int_0^t p(s)\,ds$ is bounded and if further θ satisfies the two conditions:

$$x^{-1}\theta(x) \to 0 \qquad \text{as} \quad |x| \to \infty, \tag{1.2}$$

$$x\theta(x) \geq 0 \ (\leq 0) \qquad \text{for} \quad |x| \geq h, \tag{1.3}$$

then (1.1) has at least one ω-periodic solution, for all arbitrary φ.

A careful study of Reissig's proofs will however, reveal that in the two special cases:

(i) $a > 0$ and $x\theta(x) \leq 0$ $(|x| \geq h)$

(ii) $a < 0$ and $x\theta(x) \geq 0$ $(|x| \geq h)$

or, to put it more compactly, when

$$x\theta(x) \operatorname{sgn} a \leq 0 \ (|x| \geq h) \tag{1.4}$$

(*) Pervenuta all'Accademia il 19 luglio 1974.

the existence of an ω-periodic solution can indeed be established for (1.1) without the explicit use of (1.2) on θ; and the object of the present treatment is to draw attention to other extensions of the result which are equally valid subject to the same (1.4) or to the stronger condition:

$$x\theta(x) \operatorname{sgn} a \leq -\delta x^2$$

for some constant $\delta > 0$.

We shall deal first with the equation

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + \theta(x) = p(t) \tag{1.5}$$

where ψ is continuous and depends only on $\dot{x}$, the basic conditions on φ, θ and p being the same as before.

Let $\Psi(y) \equiv \int_0^y \psi(\eta)\, d\eta$. We shall prove here

THEOREM 1. *Suppose that* $P(t)$ *is bounded for all* t, *and that there exist constants* $a \neq 0$, $h \geq 0$ *such that*

$$\Psi(y) - ay = o(1) \qquad as \quad |y| \to \infty, \tag{1.6}$$

$$x\theta(x) \operatorname{sgn} a \leq 0 \qquad (|x| \geq h). \tag{1.7}$$

Then there exists at least one ω*-periodic solution of* (1.5).

With a stronger restriction on $x\theta(x) \operatorname{sgn} a$, it is possible to extend our treatment to the perturbed system

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + \theta(x)x + \varphi(x) = p(t) + q(t, x, \dot{x}) \tag{1.8}$$

in which the basic conditions on ψ, φ, θ and p are as in (1.5 and $q(t, x, y)$ is continuous and satisfies $q(t + \omega, x, y) = q(t, x, y)$, for all t, x, y. We shall indeed prove here

THEOREM 2. *Suppose that* $P(t)$ *is bounded for all* t, *and that further:*

(i) *There exist constants* $a \neq 0$, $h \geq 0$ *and* $\delta > 0$ *such that* (1.6) *holds and such that*

$$x\theta(x) \operatorname{sgn} a \leq -\delta x^2 \ (|x| \geq h) \tag{1.9}$$

(ii) *there are constants* $\alpha \geq 0$, $\beta \geq 0$ *such that*

$$|q(t, x, y)| \leq \alpha + \beta|x| \qquad for\ all \quad t, x, y. \tag{1.10}$$

Then (1.8) *has at least one* ω*-periodic solution if* $\beta < \delta$.

Note that because the conditon (1.7) contrasts sharply with the " Routh-Hurwitz requirements ":

$$a > 0 \quad , \quad x\theta(x) > 0$$

for (1.1), there is no basis whatever for comparing the present theorems with the existence theorems (such as given in [2], [3]) for equations (1.1) of the dissipative type.

2. PROOF OF THEOREM 1. We shall deal first with the case:

$$a > 0 \qquad \text{and} \quad x\theta(x) \leq 0 \; (|x| \geq h).$$

The proof is by way of the auxiliary third order differential equation

$$\dddot{x} + \{(1 - \mu)\, a + \mu\psi(\dot{x})\}\, \ddot{x} + \mu\varphi(x)\, \dot{x} + \{(1 - \mu)\, cx + \mu\theta(x)\} = \mu p(t) \tag{2.1}$$

involving a parameter $\mu \in [0, 1]$, where $c < 0$ is an arbitrarily fixed constant. Note that, as in [1], the equation (2.1) reduces to the constant coefficient equation

$$\dddot{x} + a\ddot{x} + cx = 0$$

when $\mu = 0$ and to the equation (1.5) when $\mu = 1$. The auxiliary equation (2.1) however is not the same as the one used in [1] and its chief advantage lies in the fact that its periodic solutions can be written out easily in the form of an explicitly defined integral equation. Indeed let $\Phi(x) \equiv \int_0^x \varphi(\xi)\, ds$ and let X , F be the column vectors, and A the 3×3 matrix, given by

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} F = \begin{pmatrix} 0 \\ -\Psi(y) + ay - \Phi(x) + P(t) \\ cx - (x) \end{pmatrix}, \tag{2.2}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & 0 & 0 \end{bmatrix}.$$

Then (2.1) can be checked readily to be equivalent to the 3-vector system

$$\dot{X} = AX + \mu F(X, t). \tag{2.3}$$

Note here that $F(X, t) = F(X, t + \omega)$, since the boundedness of $P(t)$ necessarily implies that $\int_0^\omega p(t)\, dt = 0$ and therefore that $P(t)$ is ω-periodic. Also the characteristic equation for the matrix A, namely the equation

$$\lambda^3 + a\lambda^2 + c = 0$$

has no purely imaginary roots if $c \neq 0$, so that the matrix $(e^{-A\omega} - I)$ (where I is the identity 3×3 matrix) is invertible. From these it is now a straightforward matter to deduce (by, for example, adapting the argument on pp. 27–28 [4]) that

LEMMA $X = X(t)$ *is an* ω*-periodic solution of* (2.3) *if and only if* X *satisfies the integral equation*

$$X = \mu TX \tag{2.4}$$

where

$$(TX)(t) \equiv \int_t^{t+\omega} (e^{-A\omega} - I)^{-1} e^{(t-s)A} F(X(s), s)\, ds.$$

If B denotes the linear space of all ω-periodic 3-vectors $X(t)$ which are defined and are continuous for all t in $[0, \omega]$, equipped with the norm

$$\|X\| = \sup_{0 \leq t \leq \omega} (|x(t)| + |y(t)| + |z(t)|),$$

the usual arguments will show that the mapping T defined in the lemma is a completely continuous mapping of B into itself. Thus, by Schaefer's version (see [5]) of the Schauder-Tichonov fixed point theorem, the existence of a point $X \in B$ fixed under T, which corresponds in an obvious way (in view of the lemma) to an ω-periodic solution of (1.5), is assured if it can be shown that there exists a fixed positive constant D_0, independent of μ, such that

$$\|X\| \leq D_0. \tag{2.5}$$

for all solutions $X \in B$ of (2.4) with $0 < \mu < 1$.

As to the actual proof of (2.5) it will again suffice here, in view of the characterization (2.4) of all ω–periodic solutions of (2.3), to show that there exists a fixed positive constant D_1, independent of μ $(0 < \mu < 1)$, such that

$$|x(t)| \leq D_1 \tag{2.6}$$

for all ω-periodic solutions of (2.1) (with $0 < \mu < 1$) since the functions $(\Psi(y) - ay)$ and $P(t)$ appearing in the definition (2.2) of F are both bounded. The main tools for (2.6) are the two equations:

$$\int_0^{\omega} \{(1 - \mu) cx + \mu\theta(x)\}\, dt = 0 \tag{2.7}$$

$$\int_0^{\omega} [a\dot{x}^2 - \mu\dot{x}\{\Psi(\dot{x}) - a\dot{x} + P(t)\}]\, dt - \int_0^{\omega} \{(1 - \mu) cx^2 + \mu x\theta(x)\}\, dt = 0 \tag{2.8}$$

obtainable either by integrating (2.1) directly or by first multiplying (2.1) by x and then integrating. Now the fact that $c < 0$ and $x\theta(x) \leq 0$ $(|x| \geq h)$ shows, in combination with (2.7), that

$$|x(\tau)| \leq h \qquad \text{for some} \quad \tau \in [0\ \omega], \tag{2.9}$$

and at the same time enables us to derive the following estimate for the integrand in the second integral on the left hand side of (2.8):

$$(1-\mu)\,cx^2+\mu x\theta(x)\leq D_2$$

for arbitrary x and $\mu \in [0, 1]$, where $D_2 \equiv h \max_{|x|\leq h} |\theta(x)|$.

Also, since $(\Psi(y)-ay)$ and P are bounded and a is positive, the remaining integrand in (2.8) necessarily satisfies

$$a\dot{x}^2-\mu\dot{x}\{\Psi(\dot{x})-a\dot{x}+P\}\geq \frac{1}{2}a\dot{x}^2-D_3 \qquad (0\leq\mu\leq 1)$$

for some fixed D_3. Thus, by (2.8),

$$\int_0^\omega \dot{x}^2\,dt\leq 2(D_2+D_3)\,\omega a^{-1}.$$

from which, since

$$|x(t)|\leq|x(\tau)|+\left|\int_\tau^t \dot{x}\,dt\right|$$

$$\leq|x(\tau)|+\omega^{1/2}\left(\int_\tau^{\tau+\omega}\dot{x}^2\,dt\right), \qquad (\tau\leq t\leq\tau+\omega)$$

by Schwarz's inequality, we obtain, on using (2.9), that

$$|x(t)|<h+\omega\{2(D_2+D_3)\,a^{-1}\}^{1/2}, \qquad (\tau\leq t\leq\tau+\omega).$$

This estimate is the same as (2.6) since $x(t)$ is ω-periodic, and the theorem is hereby established for the case: $a>0$ and $x\theta(x)\leq 0$ $(|x|\geq h)$.

It is not necessary to write out a separate proof for the case

$$a<0 \quad , \quad x\theta(x)\geq 0 \qquad (|x|\geq h). \tag{2.10}$$

For, the substitution $t=-s$ in (1.5) reduces (1.5) to the equation

$$x'''+\bar{\psi}(x')\,x''+\varphi(x)\,x'+\bar{\theta}(x)=\bar{p}(s) \quad , \quad \left('\equiv\frac{d}{ds}\,,\,x\equiv x(-s)\right) \tag{2.11}$$

in which the functions $\bar{\psi}$, $\bar{\theta}(x)$ satisfy

$$\int_0^y \bar{\psi}(\eta)\,d\eta-a_1 y=o(1) \quad \text{as} \quad |y|\to\infty \qquad \text{and} \quad x\bar{\theta}(x)\leq 0\;(|x|\geq h) \tag{2.12}$$

where $a_1=-a>0$, if (2.10) holds, and $\int_0^t \bar{p}(s)\,ds$ is bounded if $\int_0^t p(s)\,ds$ bounded.

3. PROOF OF THEOREM 2. Again we shall treat only the case

$$a > 0 \quad \text{and} \quad x\theta(x) \leq -\delta x^2 \qquad (|x| \geq h) \tag{3.1}$$

in some detail, since the other case:

$$a < 0 \quad \text{and} \quad x\theta(x) \geq \delta x^2 \qquad (|x| \geq h)$$

can be obtained in the same way as before by means of the substitution $t = -s$ in (1.8)

The procedure in the case (3.1) is almost as in § 2, and I shall thus sketch only the outlines. The auxiliary equation and the 3-vector F, for example, are the same as before except only that p is to be replaced in the auxiliary equation by $p + q$ and the entry $cx - \theta(x)$ in F by $cx - \theta(x) + q(t, x, y)$ but the matrix A is unchanged. Next, with q subject to the restriction (1.10), it is easy to check from the form of F that an estimate such as (2.6) for all ω-periodic solutions of the corresponding auxiliary equation will again secure (2.5) and thus the existence of an ω-periodic solution of (1.8).

The two main steps in the verification of (2.6) are (2.9) and an estimate for $\int_0^\omega \dot{x}^2 \, dt$; and as before, the starting point for (2.9) is the equation

$$\int_0^\omega [(1-\mu) cx + \mu \{\theta(x) - q) - q(t, x, \dot{x})\}] \, dt = 0 \tag{3.2}$$

which one obtains by integrating the corresponding auxiliary equation from $t = 0$ to $t = \omega$. If we assume henceforth that $\beta < \delta$, then it is clear from (1.9) and (1.10) that

$$\theta(x) - q(t, x, \dot{x}) \leq 0, \qquad \text{if} \quad x \geq h_0, \tag{3.3}$$

and that

$$\theta(x) - q(t, x, \dot{x}) \geq 0, \qquad \text{if} \quad x \leq h_0, \tag{3.4}$$

where $h_0 = \max[h, \alpha(\delta - \beta)^{-1}]$. With c fixed negative as usual, it follows from (3.2), with $0 < \mu < 1$, and from (3.3) and (3.4) that

$$|x(\tau)| \leq h_0 \qquad \text{for some} \quad \tau \in [0, \omega] \tag{3.5}$$

which is the required analogue of (2.9).

It remains to obtain an estimate for $\int_0^\omega \dot{x}^2 \, dt$. We have, after multiplying the corresponding auxiliary equation by x and then integrating from $t = 0$ to $t = \omega$, that

$$\int_0^\omega [a\dot{x}^2 - \mu\dot{x}\{\Psi(\dot{x}) - a\dot{x} + P\}] \, dt - \int_0^\omega [(1-\mu) cx^2 + \mu\{x\theta(x) - xq\}] \, dt = 0. \tag{3.6}$$

As before

$$(3.7) \qquad \int_0^\omega [a\dot{x}^2 - \mu\dot{x}\{\Psi(x) - ax + P\}]\, dt \geq \frac{1}{2}\, a \int_0^\omega \dot{x}\, dt - D_2\,\omega\,.$$

Also, by (1.9) and (1.10) if $\beta < \delta$ then

$$-x\theta(x) + xq(t, x, \dot{x}) \geq -D_4$$

for some fixed constant D_4 whose magnitude depends only on h, a, β, δ and θ; and hence

$$-\int_0^\omega [(1 - \mu)\, cx^2 + \mu\{x\theta(x) + xq\}]\, dt \geq -D_4\,\omega \qquad (0 \leq \mu \leq 1).$$

The above estimate, when combined with (3.6) and (3.7) shows that

$$\int_0^\omega \dot{x}^2\, dt \leq \alpha\,(D_2 + D_4)\,\omega a^{-1}$$

and the boundedness result (2.6) can now follow as before, in view of (3.5).

This concludes the verification of the theorem.

References

[1] R. Reissig (1972) – « Ann. Mat. Pura Appl. », *92*, 199–209.
[2] J. O. C. Ezeilo (1960) – « Proc. Cambridge Philos. Soc. », *56*, 381–389.
[3] V. A. Pliss (1961) – « Dokl. Akad. Naul SSSR », *138*, 302–304.
[4] J. Hale (1953) – *Oscillations in Nonlinear Systems*, McGraw Hill, New York.
[5] H. Schaefer (1955) – « Math. Ann. », *129*, 415–416.