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Asymptotic order of solutions of $(ry')' + qy = 0$

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Equazioni differenziali ordinarie. — *Asymptotic order of solutions of* $(ry')' + qy = 0$. Nota di J. MICHAEL DOLAN e GENE A. KLAASEN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori generalizzano la norma euclidea introducendo la così detta norma D la quale consente di provare tre teoremi i quali collegano il comportamento degli integrali dell'equazione differenziale lineare del secondo ordine $(ry')' + py = 0$ col comportamento dell'integrale

$$\int_a^t \left| \frac{\mu}{r} - \frac{p}{\mu} \right| (s) \, ds \quad , \quad \mu \in C'[a, \infty].$$

Comprehensive surveys of the literature of the selfadjoint linear second-order differential equation

$$(1) \qquad (ry')' + py = 0 \qquad \text{where} \quad r > 0 \quad \text{and} \quad r, p \in C[a, \infty)$$

are real valued functions can be found in Buckley [1], Hartman [2], Swanson [3] and Willett [4]. In particular numerous papers deal with the topics of boundedness and asymptotic order of solutions of (1). The purpose of this paper is to improve certain well-known boundedness criteria for solutions of equation (1) (Barrett [6, p. 424]) by introducing a more general norm than the Euclidean norm. This technique yields a method for estimating the rate of growth of solutions of (1) and provides an improvement of a result found in Hartman [2, p. 510]. In addition, conditions are given under which no solution of (1) tends to zero at ∞.

1. INTRODUCTION OF D-NORM

The method which will be used may be stated, initially, for the vector matrix equation

$$(2) \qquad Y' = AY \,, \quad A \in C[a, \infty),$$

where A is an $n \times n$ matrix function.

The following two definitions will be needed:

DEFINITION 1. *Let* $\mathcal{D}$ *denote the set of matrices* D *such that (i)* $D \in C'[a, \infty)$ *and (ii)* D *is positive definite on* $[a, \infty)$.

(*) Nella seduta del 29 giugno 1974.

DEFINITION 2. *If* $D \in \mathcal{D}$, *then* $\|\cdot\|_D$ (*the* D-*norm*) *is defined by* $\|Y\|_D^2 = Y^T DY$, *where* Y *is a vector function in* E^n.

If Y is a nontrivial solution of (2) and $D \in \mathcal{D}$, then it is easily seen that

$$\frac{\|Y\|_D'}{\|Y\|_D} = \frac{Y^T U_D Y}{Y^T DY} \qquad \text{on} \quad [a, \infty), \tag{3}$$

where $U_D = \frac{1}{2}[A^T D + DA + D']$. (It should be observed that $\|Y\|_D > 0$ on $[a, \infty)\cdot$). Let $Z = D^{1/2} Y$, then (3) becomes

$$\frac{\|Y\|_D'}{\|Y\|_D} = \frac{Z^T R_D Z}{Z^T Z} \qquad \text{on} \quad [a, \infty) \tag{4}$$

where $R_D = D^{-1/2} U_D D^{-1/2}$ is a symmetric matrix.

The first lemma is an easy consequence of the fact that the right-hand side of (4) is a Rayleigh Quotient and the fact that the j-th eigenvalue function of R_D is continuous (Lancaster [7, p. 30]).

LEMMA 1. *If* λ_D^+ *and* λ_D^-, *respectively, denote the maximum and minimum eigenvalue function for* R_D, $D \in \mathcal{D}$, *then* λ_D^+ *and* λ_D^- *are continuous on* $[a, \infty)$ *and*

$$\|Y\|_D(a) \exp\left[\int_a^t \lambda_D^-(s)\, ds\right] \le \|Y\|_D(t) \le \tag{5}$$

$$\le \|Y\|_D(a) \exp\left[\int_a^t \lambda_D^+(s)\, ds\right], \qquad t \ge a.$$

2. THE D-NORM AND SECOND ORDER EQUATIONS

We wish to apply Lemma 1 to equation (1) with a properly chosen matrix D. It is clear that equation (1) takes the form (2) if

$$Y = \begin{pmatrix} y \\ ry' \end{pmatrix} \qquad \text{and} \quad A = \begin{pmatrix} 0 & \frac{1}{r} \\ -p & 0 \end{pmatrix}.$$

Let $D = \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}$ where $\mu > 0$ and is continuously differentiable on $[a, \infty)$, then the following lemma gives estimates of bounds for the corresponding eigenvalue functions λ_D^+ and λ_D^- of the matrix R_D.

LEMMA 2. *If* $\mu \in C'[a,\infty)$ *is positive and if* $\lambda_D^-(t)$ *and* $\lambda_D^+(t)$ *are respectively the minimum and maximum eigenvalue of* $R_D(t)$ *for each* $t \in [a,\infty)$ *then*

$$\text{(6)}\qquad \begin{aligned} &\lambda_D^- \geq \frac{1}{2}\left[\frac{\mu'}{\mu} - \left|\frac{\mu'}{\mu}\right| - \left|\frac{\mu}{r} - \frac{p}{\mu}\right|\right] \quad \text{and} \\ &\lambda_D^+ \leq \frac{1}{2}\left[\frac{\mu'}{\mu} + \left|\frac{\mu'}{\mu}\right| + \left|\frac{\mu}{r} - \frac{p}{\mu}\right|\right] \quad \text{on} \quad [a,\infty]. \end{aligned}$$

Proof. By definition,

$$R_D = D^{-1/2} U_D D^{-1/2} = \frac{1}{2} D^{-1/2}[A^T D + DA + D'] D^{-1/2} =$$

$$= \frac{1}{2}\begin{pmatrix} \frac{1}{\mu} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2\mu\mu' & -p(1+\mu^2) \\ \frac{1}{r}(1+\mu^2) & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\mu} & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{2\mu'}{\mu} & \frac{\mu}{r} - \frac{p}{\mu} \\ \frac{\mu}{r} - \frac{p}{\mu} & 0 \end{pmatrix}.$$

Consequently, the eigenvalues of R_D are $\lambda = \frac{1}{2}\left[\frac{\mu'}{\mu} \pm \sqrt{\left(\frac{\mu'}{\mu}\right)^2 + \left(\frac{\mu}{r} - \frac{p}{\mu}\right)^2}\right]$ from which (6) follows immediately.

Lemma 1 and Lemma 2 are now used to prove the main theorem.

THEOREM 1. *Let* $\mu \in C'[a,\infty)$ *be a positive function on* $[a,\infty)$ *and* y *be any nontrivial solution of* (1).

(i) *If* $\mu' \leq 0$ *on* $[a,\infty)$ *then there exists constants* $m > 0$ *and* $M > 0$ *such that* $m \exp\left(-\int_a^t \left|\frac{\mu}{r} - \frac{p}{\mu}\right|(s)\,ds\right) \leq \left[y^2 + \left(\frac{ry'}{\mu}\right)^2\right](t)$ *and*

$$[(\mu y)^2 + (ry')^2](t) \leq M \exp\left(\int_a^t \left|\frac{\mu}{r} - \frac{p}{\mu}\right|(s)\,ds\right) \quad \textit{on } [a,\infty).$$

(ii) *If* $\mu' \geq 0$ *then there exists* $m > 0$ *and* $M > 0$ *such that*

$$m \exp\left(-\int_a^t \left|\frac{\mu}{r} - \frac{p}{\mu}\right|(s)\,ds\right) \leq [(\mu y)^2 + (ry')^2](t) \quad \textit{and}$$

$$\left[y^2 + \left(\frac{ry'}{\mu}\right)^2\right](t) \leq M \exp\left(\int_a^t \left|\frac{\mu}{r} - \frac{p}{\mu}\right|(s)\,ds\right) \quad \text{on } [a,\infty).$$

Proof. $D = \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}$ implies that $\|Y\|_D^2 = (\mu y)^2 + (ry')^2$.

Considering the case $\mu' \leq 0$, Lemma 2 implies that $\lambda_D^- \geq \frac{\mu'}{\mu} - \frac{1}{2}\left|\frac{\mu}{r} - \frac{\mu}{\mu}\right|$ and $\lambda_D^+ \leq \frac{1}{2}\left|\frac{\mu}{r} - \frac{p}{\mu}\right|$. Lemma 1 then yields:

$$\|Y\|_D^2(a)\left(\frac{\mu^2(t)}{\mu^2(a)}\right)\exp\left(-\int_a^t \left|\frac{\mu}{r} - \frac{p}{\mu}\right|(s)\,ds\right) \leq [(\mu y)^2 + (ry')^2](t) \leq$$

$$\leq \|Y\|_D^2(a) \exp \int_a^t \left|\frac{\mu}{r} - \frac{p}{\mu}\right|(s)\,ds\,.$$

Conclusion (i) of Theorem 1 is obtained from these inequalities where we choose $m = \frac{\| Y \|_D^2 (a)}{\mu^2 (a)}$ and $M = \| Y \|_D^2 (a)$. A similar argument will yield conclusion (ii) of this theorem.

One might expect more generality in Theorem 1 if the matrix D is allowed to take the more general diagonal form $D = \begin{pmatrix} \varphi^2 & 0 \\ 0 & \psi^2 \end{pmatrix}$. This is however not the case since $\frac{1}{\psi^2}$ D is of the form used in Theorem 1 and the scalar coefficient $\frac{1}{\psi^2}$ plays no role in the theory. If $\begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}$ is replaced by a positive definite but not diagonal matrix in the above discussion then two complications arise. Under these circumstances $D^{-1/2}$, and λ_D^+ and λ_D^- are difficult to compute and moreover $\| Y \|_D$ takes a form which includes a term of the form $gyry'$ for some function g determined by D. The presence of this term requires a more careful analysis and will not be considered in this paper.

The following corollaries yield information about the asymptotic order and boundedness of solutions of (1).

COROLLARY 1. *Suppose* $\left(\frac{\mu}{r} - \frac{p}{\mu}\right) \in L_1 [a, \infty)$ *for some positive function* $\mu \in C' [a, \infty)$ *and let* y *be any nontrivial solution of* (1).

(i) *If* $\mu' \leq 0$, *then as* $t \to \infty$ $y = O(\mu^{-1})$ *and* $ry' = O(1)$ *and there exists an* $m > 0$ *such that* $y^2 + \left(\frac{ry'}{\mu}\right)^2 \geq m$ *on* $[a, \infty)$.

(ii) *If* $\mu' \geq 0$, *then as* $t \to \infty$ $y = O(1)$ *and* $ry' = O(\mu)$ *and there exists an* $m > 0$ *such that* $(\mu y)^2 + (ry')^2 \geq m$ *on* $[a, \infty)$.

This result follows immediately from Theorem 1 and needs no proof.

Corollary 1 can be considered a generalization of a result of Barrett [6, p. 424], viz., if for some positive number h, it is true that $\left[\frac{h}{r} - \frac{p}{h}\right] \in L_1 [a, \infty)$ then $y = O(1)$ and $ry' = O(1)$ at ∞ for each solution of (1).

The following corollary is a generalization of a result stated in Loud [5] and credited to Liapanhoff, viz., if $A(t)$ is defined bounded and piecewise continuous for $t \geq a$ and x_0 is a nontrivial vector solution of $x'(t) = A(t)\, x(t)$, $t \geq a$, then there is a number λ_0 such that for $\lambda > \lambda_0$, $\lim_{t \to \infty} x_0(t)\, e^{-\lambda t} = 0$.

COROLLARY 2. *If for some number* λ, $\frac{e^{\lambda t}}{r(t)} - \frac{p(t)}{e^{\lambda t}} \leq K$ *on* $[a, \infty)$ *where* K *is a positive constant, then for each nontrivial solution* y *of* (1);

(i) $y = O(e^{\alpha t})$ *for* $\alpha \geq \frac{K}{2}$ *and* $ry' = O(e^{\beta t})$ *for* $\beta \geq \frac{K}{2} + \lambda$ *whenever* $\lambda \geq 0$ *and*

(ii) $y = O(e^{\alpha t})$ *for* $\alpha \geq \frac{K}{2} - \lambda$ *and* $ry' = O(e^{\beta t})$ *for* $\beta \geq \frac{K}{2}$ *whenever* $\lambda \leq 0$.

Proof. Only conclusion (i) will be proved since conclusion (ii) is proved by a similar argument. Suppose $\lambda \geq 0$ and let $\mu = e^{\lambda t}$, then $\mu' \geq 0$. It follows from conclusion (ii) of Theorem 1 that for some $M > 0$,

$$\left[y^2 + \left(\frac{ry'}{\mu}\right)^2\right](t) \leq M \exp\left(\int_a^t \left|\frac{\mu}{r} - \frac{p}{\mu}\right|(s)\, ds\right) \qquad \text{on} \quad [a, \infty).$$

The integrand above is bounded by K and hence $y^2 \leq M e^{K(t-a)}$ and $\left(\frac{ry'}{e^{\lambda t}}\right)^2 \leq M e^{K(t-a)}$ from which follows conclusion (i).

The next theorem generalizes some well-known results (Bellman [8, p. 111], Hartment [2, p. 510], Leighton [9]).

THEOREM 2. *Suppose* $p > 0$, $(pr)' \geq 0 \{(pr)' \leq 0\}$ *on* $[a, \infty)$ *and* $\frac{q}{\sqrt{pr}} \in L_1[a, \infty)$. *If y is any nontrivial solution of* $(ry')' + (p+q)y = 0$ *on* $[a, \infty)$ *then* $y = O(1)$ *and* $ry' = O(\sqrt{pr})$ $\left\{y = O\left(\frac{1}{\sqrt{pr}}\right)\right.$ *and* $\left.ry' = O(1)\right\}$ *at* ∞.

Proof. Making use of Corollary 1 and choosing $\mu = \sqrt{pr}$ we notice that $(pr)' \geq 0$ implies that $\mu' \geq 0$. Also $\frac{\mu}{r} - \frac{p+q}{\mu} = \frac{q}{\sqrt{pr}} \in L_1[a, \infty)$ and hence $y = O(1)$ and $ry' = O(\sqrt{pr})$. A similar argument covers the bracketed case.

Some of the references in the introduction of this paper give sufficient conditions for at least one solution of equation (1) to have limit zero at ∞. Corollary 1 may be used to state sufficient conditions that no solution tends to zero at ∞. These sufficient conditions are given in terms of the oscillatory behavior of (1). See Barrett [6] or Swanson [3] for a discussion of oscillation theory.

THEOREM 3. *If* $\left[\frac{\mu}{r} - \frac{p}{\mu}\right] \in L_1[a, \infty)$ *where* $\mu > 0$ $\mu' \leq 0$ *on* $[a, \infty)$, *then* (i) *no solution of* (1) *tends to zero at* ∞ *if equation* (1) *is oscillatory, and* (ii) *no solution of* (1) *tends to zero at* ∞ *if equation* (1) *is nonoscillatory and* $\frac{\mu}{r} \notin L_1[b, \infty)$ *for* $b \geq a$.

Proof. If y is a solution of (1), then by part (i) of Corollary 1 there exists $k > 0$ such that $0 < k \leq \left[y^2 + \left(\frac{ry'}{\mu}\right)^2\right](t)$ for $t \geq a$. Suppose, contrary to (i), that there is an oscillatory solution y of (1) such that $\lim_{t\to\infty} y(t) = 0$, then there is a number $b \geq a$ such that $\left(\frac{ry'}{\mu}\right)^2 \geq \frac{1}{2}k > 0$ on $[b, \infty)$. This contradicts the oscillatory behavior of y and y'.

Secondly, contrary to (ii), suppose y is an eventually positive solution of (1) such that $\lim_{t\to\infty} y(t) = 0$. Let $c \geq a$ be such that $\left[\frac{ry'}{\mu}\right]^2(t) \geq \frac{1}{2}k > 0$ for $t \geq c$, then $y \leq -\frac{k\mu}{2r}$ eventually and $y(t) \leq y(c) - \frac{k}{2}\int_c^t \frac{\mu}{r}(s)\, ds$.

But $\frac{\mu}{r} \notin L_1[b, \infty)$ implies that $y(t)$ becomes negatively unbounded for large t. This contradiction forces the conclusion that no nontrivial solution has a zero limit.

By an appropriate choice of D, it is possible that the D-norm of solutions of (1) might be bounded but the Euclidean norm, $\| Y \|^2 = y^2 + (ry')^2$, might be unbounded. Hence the D-norm can give more information about the boundedness of solutions in that case. The following example will indicate such a situaton.

Let $r(t) = t^{2\alpha}$ and $p(t) = t^{2\alpha}[1 + \alpha(\alpha - 1)t^{-2}]$ in equation (1) where $[a, \infty) = [1, \infty)$. If $\mu(t) = t^{2\alpha}[1 + \alpha(\alpha - 1)t^{-2}]^{-1/2}$ for $t \geq 1$ then $\frac{\mu}{r} - \frac{p}{\mu} = 0$ and for sufficiently large t, it can be shown that $\mu > 0$ and $\operatorname{sgn} \mu' = \operatorname{sgn} \alpha$. Using Corollary 1 we obtain the following information.

If $\alpha < 0$ then $y = O(t^{-2\alpha})$ and $ry' = O(1)$ as $t \to \infty$.
If $\alpha = 0$ then $y = O(1)$ and $ry' = O(1)$ as $t \to \infty$.
If $\alpha > 0$ then $y = O(1)$ and $ry' = O(t^{2\alpha})$ as $t \to \infty$.

Thus if $\alpha \geq 0$ we conclude that all solutions of (1) are bounded. However, since the general solutions of (1) is $y(t) = t^{-\alpha}(A \sin t + B \cos t)$, $y^2 + (ry')^2 = O(t^{|\alpha|})$ and hence the Euclidean norm is unbounded if $\alpha > 0$.

In conclusion it is noted that one can relax the continuity hypothesis on p and r somewhat and still obtain the results of Theorem 1. All that is required is that λ_D^+, λ_D^- and $\left| \frac{\mu}{r} - \frac{p}{\mu} \right|$ are locally integrable on $[a, \infty)$.

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