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Symbolic calculus in $A_p(G)$. Nota II

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Analisi matematica. — *Symbolic calculus in* $A_p(G)$. Nota II (*) di LEONEDE DE MICHELE e PAOLO SOARDI, presentata dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dimostra che solo le funzioni reali analitiche operano nell'algebra $A_p(G)$ se G è un gruppo abeliano infinito, oppure un gruppo mediabile discreto con sottogruppi abeliani arbitrariamente grandi, oppure un gruppo di Lie non discreto.

4. THE ABELIAN CASE; NONDISCRETE GROUPS

In this section we need the following results of I. Khalil ([7], Theorem 5, Lemma 7),

PROPOSITION 1. *Let* $I \subseteq \mathbf{T}$ *be a closed interval and* φ *a continuous function on* I; *let* C *be a positive number. Then the following statements are equivalent:*

1) *for ever finitely supported measure* μ *on* $\mathbf{T}$ *supported by* I

$$\left| \int\limits_{E} \varphi(t) \, d\mu \right| \leq C \, \| \mu \|_{Cv_p(\mathbf{T})}$$

2) *there exists* $\psi \in A_p(\mathbf{T})$, $\|\psi\|_{A_p(\mathbf{T})} \leq C$, *such that for every* $x \in I$ $\varphi(x) = \psi(x)$.

PROPOSITION 2. *Let* μ *be a Radon measure on the real line* $\mathbf{R}$, *supported by a closed interval* $I \subseteq [-1/10, 1/10]$. *Let* $\dot{\mu}$ *be the Radon measure on* $\mathbf{T}$ *defined by*

$$\int\limits_{\mathbf{T}} \varphi(t) \, d\dot{\mu}(t) = \int\limits_{\mathbf{R}} \varphi(e^{ix}) \, d\mu(x)$$

for every φ *continuous on* $\mathbf{T}$. *Then*:

$$\| \mu \|_{Cv_p(\mathbf{R})} \leq 2 \, \| \dot{\mu} \|_{Cv_p(\mathbf{T})} .$$

For every complex-valued function f supported in $(-\pi, \pi)$ we shall denote by $s(f)$ the function on $\mathbf{T}$ defined by

$$s(f)(e^{ix}) = \sum_{n=-\infty}^{+\infty} f(x + 2\pi n)$$

for all $x \in \mathbf{R}$. The next theorem is a consequence of Khalil's results.

THEOREM 2. *Let* f *be a complex-valued function defined on* $\mathbf{R}$ *with support in* $(-\pi, \pi)$. *The following statements are equivalent*:

1) $f \in A_p(\mathbf{R})$.

2) $s(f) \in A_p(\mathbf{T})$.

(*) Pervenuta all'Accademia il 19 luglio 1974.

Proof. 2) implies 1). Indeed, by [4], Theorem 1, $s(f)(e^{ix})$ belongs to $B_p(\mathbf{R})$. Let $h \in A_p(\mathbf{R})$ be a function such that supp $h \subseteq (-\pi, \pi)$ and $h(x) = 1$ for every $x \in \operatorname{supp} f$. Then $f(x) = h(x)\, s(f)(e^{ix})$ for all x and so $f \in A_p(\mathbf{R})$.

1) implies 2). It suffices to prove that $s(f)$ belongs to $A_p(\mathbf{T})$ locally at every point of $\mathbf{T}$ (see [9], Ch. 6). At first we prove that $s(f)$ belongs to $A_p(\mathbf{T})$ locally at zero. Let $h \in A_p(\mathbf{R})$ such that supp $h \subseteq [-1/10, 1/10]$, $0 \leq h(x) \leq 1$, $h(x) = 1$ in a neighborhood of zero.

Then $s(hf) \in A_p(\mathbf{T})$. Indeed, there exists $C > 0$ such that, for every Radon measure μ on $\mathbf{R}$ with finite support contained in $[-1/10, 1/10]$ and $\|\mu\|_{Cv_p(\mathbf{R})} \leq 1$ we have:

$$\left| \int_{\mathbf{R}} h(x) f(x)\, d\mu(x) \right| \leq C .$$

By Proposition 2

$$\operatorname{Sup} \left| \int_{\mathbf{T}} s(hf)(t)\, d\dot{\mu}(t) \right| \leq 2C$$

where the supremum is taken over all Radon measures with finite support contained in the natural image of $[-1/10, 1/10]$ in $\mathbf{T}$ and $Cv_p(\mathbf{T})$-norm not larger than 1. By Proposition 1 $s(hf) \in A_p(\mathbf{T})$; consequently $s(f)$ belongs to $A_p(\mathbf{T})$ locally at zero.

Now, trivially, $s(f) \in A_p(\mathbf{T})$ locally at π; by translation it is easily seen that $s(f)$ belongs to $A_p(\mathbf{T})$ locally at every point of $\mathbf{T}$.

THEOREM 3. *Let* G *be an infinite nondiscrete abelian group. Let* F *be a complex-valued function defined in a closed convex subset* E *of* $\mathbf{C}$. *Then* F *operates in* $A_p(G)$ *if and only if* F *is real-analytic in* E. *Moreover* $F(0) = 0$ *if* G *is noncompact.*

Proof. It is well known ([9], 2.4.1.) that G contains an infinite compact group or a closed subgroup isomorphic to the real line $\mathbf{R}$.

In the first case, since $A_p(G)|_H = A_p(H)$ for every closed subgroup H, the theorem follows from Theorem 1. In the second case, Theorem 2 allows us to apply the same arguments as in [9] 6.6.4.

5. THE DISCRETE AMENABLE CASE

Through this section G will be an infinite discrete amenable group; we shall suppose, for the sake of simplicity, that the function F, operating in $A_p(G)$, is defined in $[-1, 1]$. Many arguments of this section are the same as in our previous paper [3].

LEMMA 4. *There are positive numbers* δ *and* M *such that for every* $f \in B_p(G)$ *with* $\|f\|_{B_p(G)} \leq \delta$ *it follows*

$$F(f) \in B_p(G) \tag{5.1}$$

$$\| F(f) \|_{B_p(G)} \leq M . \tag{5.2}$$

Proof. Since G is amenable, there is in $A_p(G)$ a bounded approximate unit. Then it is easy to see, just as in [3], Proposition 1, that the statement is true for finitely supported functions. Then, let $f \in B_p(G)$, $\|f\|_{B_p(G)} \leq \delta/2$. For every finite set $K \subseteq G$ it is possible to find a finitely supported function v_K such that $v_K(x) = 1$ if $x \in K$ and $\| v_K \|_{B_p(G)} \leq 2$.

Therefore

$$F(f)(x) = \lim_{K} F(fv_K)(x) \tag{5.3}$$

for all $x \in G$, and

$$\| F(fv_K) \|_{B_p(G)} \leq M. \tag{5.4}$$

Since G is discrete and the unit ball of B_p is closed for the compact convergence ([4], 3.2), (5.1) and (5.2) follow from (5.3) and (5.4).

LEMMA 5. *If* G *contains abelian subgroups of arbitrarily large order, then there exists* $b > 0$ *such that, for sufficiently large* r:

$$\operatorname*{Sup}_{f \in S_r} \| e^{if} \|_{B_p(G)} \geq e^{br}. \tag{5.5}$$

Proof. First suppose that G is not of bounded exponent. Then G contains an element of infinite order or arbitrarily large cyclic finite subgroups.

In the first case $G \supseteq \mathbf{Z}$ (the relative integers). Since only real-analytic functions operate in $A_p(\mathbf{R})$, (5.5) is necessarily true if $G = \mathbf{R}$ (see [9], 6.7.1.).

Notice that I. Khalil proved ([7], Theorem 1) that a function u defined and continuous in $\mathbf{R}$ belongs to $B_p(\mathbf{R})$ and $\| u \|_{B_p(\mathbf{R})} \leq C$ if and only if for all $\lambda > 0$ $\{u(\lambda n)\}_{n=-\infty}^{+\infty}$ belongs to $B_p(\mathbf{Z})$ with norm not larger than C. Denote now by S'_r the set of the real-valued functions in $B_p(\mathbf{Z})$ with norm not larger than r. By Khalil's result:

$$\operatorname*{Sup}_{f \in S'_r} \| e^{if} \|_{B_p(\mathbf{Z})} \geq e^{br}. \tag{5.6}$$

Since there is in $A_p(\mathbf{Z})$ a bounded approximate unit, we get (5.6) with S_r instead of S'_r. Since the restriction is a norm decreasing application of $B_p(G)$ into $B_p(\mathbf{Z})$ ([4], p. 59), (5.5) is true.

If G contains finite cyclic subgroups of arbitrarily large order G_r, then, since G_r is compact, it is possible, by Lemmas 1 and 2, to repeat the first part of the proof of Theorem 1 for such groups.

Finally, let G be of bounded exponent; then G must contain arbitrarily large abelian subgroups with the same finite exponent. Therefore, by [6]

I.13.12, there exists a prime number q such that G contains abelian subgroups of order q^{α}, with α arbitrarily large. It is again possible to repeat the second part of the Theorem 1.

Remark. We have incidentally proved, for the groups we are dealing with, that $B_p(G) \neq L^{\infty}(G)$.

LEMMA 6. *Let* G *be as in Lemma 5. Then there exists a set* $\Lambda \subseteq G$ *such that if* $f \in B_p(G)$ *and supp* $f \subseteq \Lambda$, *then* $\inf_{\Lambda} |f(x)| = 0$.

Proof. Assume, by way of contradiction, that the lemma is false. Then, by the same technique as in [3], lemma, it is possible to approximate in $L^{\infty}(G)$-norm the characteristic function of every subset of G; by theorem 3.3 of [2], applied to the Stone-Čech compactification of G, one gets $B_p(G) = L^{\infty}(G)$, which is absurd by the previous remark.

Now the same proof as in [3], Proposition 3, gives:

LEMMA 7. *Let* G *satisfy the same hypothesis as in Lemma 5; then* F *is continuous in a neighborhood of zero.*

THEOREM 4. *Let* G *be a discrete amenable group containing abelian subgroups of arbitrarily large order. Let* F *be a complex-valued function defined in* $[-1, 1]$. *Then* F *operates in* $A_p(G)$ *if and only if* F *is real-analytic in a neighborhood of zero and* $F(0) = 0$.

Proof. It suffices to remark that the foregoing lemmas are all that is needed in order to apply the proof of Helson, Kahane, Katznelson and Rudin [5]

Remark. The assumptions of Theorem 4 are satisfied, for instance, by the following groups:

1) amenable groups containing an infinite abelian subgroup (in particular infinite abelian groups);

2) amenable groups of unbounded exponent;

3) amenable groups of bounded exponent containing arbitrarily large finite subgroups;

4) amenable infinite groups having as exponent one of the following numbers: 2, 3, 4, 6.

1) and 2) are trivial; in order to establish 3) remark that (see [8], proof of Proposition 4), by Sylow's theorem, the finite arbitrarily large subgroups of a group of bounded exponent contain large q-groups for some prime number q. On the other hand, a q-group of order q^n contains an abelian group of order q^{α}, where $\alpha(\alpha+1) \geq 2n$ (see [6], III.7.3).

To establish 4) remark that Burnside's conjecture is true for groups of exponent 2, 3, 4, 6 (see [6], III.6.7.).

Therefore such groups are locally finite and so they contain arbitrarily large finite subgroups.

Notice that Burnside's conjecture is not true in general; there are groups of sufficiently high exponent which are not locally finite [1]. However, the Authors do not know examples of infinite groups not containing abelian subgroups of arbitrarily large order.

6. THE NONAMENABLE CASE; LIE GROUPS

Let us now consider functions operating in the A_p algebra of a not necessarily amenable locally compact group like, for instance, Lie groups. Our starting point is the following: if G contains an infinite closed subgroup H satisfying the assumptions of one of the previous theorems, then, since $A_p(G)|_H = A_p(H)$, only real-analytic functions operate in $A_p(G)$. Suppose, for instance that G contains an infinite abelian subgroup; then one gets:

THEOREM 5. *Let* G *be a locally compact group containing an infinite abelian subgroup* H, *and let* F *be a complex-valued function defined in* $[-1, 1]$. *Then, if the closure of* H *is nondiscrete,* F *operates in* $A_p(G)$ *if and only if it is real-analytic in* $[-1, 1]$, *and* $F(0) = 0$ *if* G *is noncompact. If* G *is discrete, then* F *operates in* $A_p(G)$ *if and only if it is real-analytic in a neighborhood of zero and* $F(0) = 0$.

Remark. By this theorem only real-analitic functions operate in the algebra A_p of a nondiscrete Lie group, since it contains nondiscrete one-parameter subgroups. By the same theorem, if G is the discrete free group with two generators, a complex-valued function operates in $A_p(G)$ if and only if it is real-analytic in a neighborhood of zero and $F(0) = 0$.

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