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Analisi matematica. — *Symbolic calculus in* $A_p(G)$. Nota I [(*)]
di LEONEDE DE MICHELE e PAOLO SOARDI, presentata dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dimostra che solo le funzioni reali analitiche operano nell'algebra $A_p(G)$ se G è un gruppo compatto.

1. INTRODUCTION

Let G be a locally compact group and $L^p(G)$ the Lebesgue space corresponding to the left-invariant Haar measure on G; we denote by $\|\cdot\|_p$ the norm in $L^p(G)$ and by p' the conjugate exponent of p. It is well known [10] that the set $A_p(G)$ $(1 < p < \infty)$ of all functions u on G of the form

$$u = \sum_{n=1}^{\infty} f_n * \check{g}_n \tag{1.1}$$

$\left(f_n \in L^p(G), g_n \in L^{p'}(G), \check{g}_n(x) = g_n(x^{-1}), \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'} < \infty\right)$ is a Banach algebra under pointwise multiplication with the norm:

$$\|u\|_{A_p(G)} = \inf \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'}$$

where the infimum is taken over all possible representations (1.1) of u. When $p = 2$, $A_p(G) = A(G)$, the Fourier algebra of G (see [4]). If $L^p \hat{\otimes} L^{p'}$ is the Banach space tensor product of L^p and $L^{p'}$, and if P is the application of $L^p \hat{\otimes} L^{p'}$ into $C_0(G)$ defined by $P(f \otimes g) = f * \check{g}$, $A_p(G)$ may be identified to the quotient space $L^p \hat{\otimes} L^{p'}/\mathrm{Ker}\, P$, and the norm $\|\cdot\|_{A_p}$ is the quotient norm.

As first proved A. Figà-Talamanca ([6], [7]), $A_p(G)$ is related by duality to the convolutors of $L^p(G)$; if G is amenable, the dual space of $A_p(G)$ is isometrically isomorphic to $Cv_p(G)$, the L^p-convolutors space, by the relation:

$$\langle u, T \rangle = \sum_{n=1}^{\infty} \langle Tf_n, g_n \rangle$$

where $T \in Cv_p(G)$ and u is as in (1.1) (see [4], [6], [7], [13]). If $p = 2$, according to Eymard's notations [4], $Cv_2(G)$ will be denoted by VN (G).

Let $B_p(G)$ denote the set of all bounded, continuous, complex-valued functions on G which are multipliers of $A_p(G)$. $B_p(G)$ is a Banach algebra for the pointwise multiplication wih the usual operator norm. If G is amenable

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$A_p(G)$ has a bounded approximate unit (see [5], theorem 3), and the injection of $A_p(G)$ in $B_p(G)$ is actually an isometry; $B_p(G)$ may also be viewed as the dual space of $L^1(G)$ with the L^p-convolutors norm. The duality is given (see Herz [12]) by

$$\langle u, f\rangle = \int_G f(x)\, u(x)\, dx$$

for all $f \in L^1(G)$ and $u \in B_p(G)$.

When G is amenable $B_2(G) = B(G)$, the Fourier-Stieltjes algebra of G; moreover one gets the following continuous injections (see [13]):

$$M(G) \subsetneq Cv_r(G) \subsetneq Cv_p(G) \subsetneq VN(G)$$

$$A(G) \subsetneq A_r(G) \subsetneq A_p(G) \subsetneq C_0(G)$$

if $1 < p \leq r \leq 2$ or $2 \leq r \leq p$.

In this paper we shall often use the following result due to C. Herz [11]: if H is a closed subgroup of a locally compact group G, then the restriction $A_p(G)|_H$ of $A_p(G)$ to H is exactly $A_p(H)$; moreover $\|f|_H\|_{A_p(H)} \leq \|f\|_{A_p(G)}$. If $p = 2$, for all $g \in A(H)$ there is $f \in A(G)$ such that $f|_H = g$ and $\|f\|_{A(G)} = \|g\|_{A(H)}$.

We say that a complex-valued function F, defined on a subset E of the complex plane **C**, operates in $A_p(G)$ if the composition $F(f)$ belongs to $A_p(G)$ whenever $f \in A_p(G)$ and the range of f is in E. If $p = 2$, E is closed and convex and G is an infinite nondiscrete abelian group it is well known [9] that F operates in $A(G)$ if and only if F is real analytic in E (and $F(0) = 0$ if G is noncompact). If G is an infinite discrete abelian group and E is a neighborhood of the origin, F operates in $A(G)$ if and only if F is real analytic in a neighborhood of zero and $F(0) = 0$. These results have been extended to a large class of noncommutative locally compact groups (see [1], [3], [16]). If $p \neq 2$, the only results the authors know on this subject are due to Drury [2] and Fisher [8]. Drury proved that only real-analytic functions operate in $A_p(G)$ when G is the 1-dimensional torus **T** or the Cantor group; Fisher proved, by different techniques, the same result for any compact abelian group. In this paper we start from the works of Drury and Rider [16]; by sharpening their techniques we are able to prove that only real-analytic functions operate in $A_p(G)$ if:

1) G is an infinite abelian group,

2) G is an infinite compact group,

3) G is an infinite discrete amenable group containing arbitrarily large abelian subgroups;

4) G is a nondiscrete Lie group (see also [15]).

Therefore the results obtained for $p = 2$ in [9] and [16] are extended to the general case. The result 3) is new also for $p = 2$.

2. THE COMPACT CASE: PRELIMINARY LEMMAS

Let p be a fixed number, $1 < p < \infty$. Then

LEMMA 1. *Let* G *be a locally compact amenable group with the following property: there exists* $b > 0$ *such that for some positive number* r *there is a compact subgroup* $G_r \subseteq G$ *and a real-valued function* $f_r \in A_p(G)$ *such that*:

$$\| f_r \|_{A_p(G)} \leq r$$

$$\| e^{-if_r|G_r} \|_{Cv_p(G_r)} \leq e^{-br}$$

Then

$$\sup_{f \in S_r} \| e^{if} \|_{B_p(G)} \geq e^{br} \tag{2.1}$$

where S_r *is the set of all real-valued functions* $f \in A_p(G)$ *of norm not larger than* r.

Proof. Since the application $B_p(G) \to A_p(G_r)$ is norm-decreasing (see [5], p. 59. Remark 2) we have, by duality:

$$1 = \langle e^{if_r|G_r}, e^{-if_r|G_r} \rangle \leq \| e^{if_r|G_r} \|_{A_p(G_r)} \| e^{-if_r|G_r} \|_{Cv_p(G_r)} \leq \| e^{if_r} \|_{B_p(G)} \cdot e^{-br}.$$

Hence $\| e^{if_r} \|_{B_p(G)} \geq e^{br}$, and (2.1) follows.

LEMMA 2. *Let* G *be a compact group and* μ *a Radon measure on* G *of norm* 1. *Then*

$$\| \mu \|_{Cv_p(G)} \leq \| \mu \|_{VN(G)}^{\sigma} \tag{2.2}$$

where $\sigma = 2/p'$ *if* $1 < p \leq 2$, *and* $\sigma = 2/p$ *if* $2 \leq p < \infty$.

Proof. The inequality (2.2) is a straightforward consequence of the Riesz-Thorin theorem, by interpolating between $L^1(G)$ and $L^2(G)$ if $1 < p \leq 2$, and between $L^2(G)$ and $L^\infty(G)$ if $2 \leq p < \infty$.

LEMMA 3. *Let* H *be a finite abelian group of order* q^α *and exponent* q^β *where* q *is a prime number. If* $\alpha > \beta N$ *for some integer* N, *then* H *contains at least* N *independent elements of order* q.

Proof. On account of a well known theorem (see, for instance, [14] I.13.12) there are n cyclic subgroups H_i of order $q^{\gamma_i} (i = 1, \cdots, n)$ with $1 \leq \gamma_i \leq \beta$, such that H is isomorphic to the direct product of $H_1, \cdots, H_n$. Since

$$q^\alpha = q^{\sum_{i=1}^{n} \gamma_i} \leq q^{n\beta}$$

it follows $n > N$. If h_i is a generator of H_i, the elements $h_i^{q^{\gamma_i - 1}}$ are n independent elements of order q.

3. THE COMPACT CASE: THE MAIN RESULT

THEOREM 1. *Let* G *be an infinite compact group and* F *a complex-valued function defined in a closed convex subset* E *of* **C**. *Then* F *operates in* $A_p(G)$ *if and only if* F *is real-analytic in* E.

Proof. Since $A_p(G)$ is a regular symmetric algebra, we have only to show, in order to apply the classical proof of Helson, Kahane, Katznelson and Rudin ([9]; see also [17] ch. 6), that there is $b > 0$ such that (2.1) holds when r is sufficiently large. We divide the proof into several steps.

1) G has no bounded exponent. Then either of two is true:

a) for every positive integer S there is a cyclic finite subgroup H of order larger than S and, consequently, there is a character on H of order larger than S;

b) G contains a closed infinite monothetic subgroup H and so the dual of H has not bounded exponent ([17], 2.33).

In both cases there are closed abelian subgroups of G with continuous characters of arbitrarily large order. This allows us to generalize an argument of Drury.

Let $J_n(x)$ be the n-th Bessel function (n relative integer); there is a real number a such that

$$0 < a < 1/2 \tag{3.1}$$

and

$$\sum_{n\neq 0} |J_n(a)| < J_0(a) < 1 \tag{3.2}$$

(see, for instance [18], p. 16).
Let us set $j_n = J_n(a)$; there exists $n_0 > 0$ such that

$$\sum_{|n|\geq n_0} |j_n| \leqq (1 - j_0)/2 .$$

Let $\{\lambda_n\}$ be a sequence of positive integers such that

$$\lambda_{n+1} = 2 n_0 \lambda_n \qquad n = 0, 1, \cdots; \tag{3.3}$$

for every $r \geq 1$ Let $N = [r] + 1$ and

$$S = \sum_{n=0}^{N} n_0 \lambda_n = 2 \lambda_0 n_0 \sum_{n=0}^{N} (2 n_0)^n . \tag{3.4}$$

By the foregoing remark, there exists an abelian closed subgroup $G_r \subseteq G$ with a continuous character φ_r of order larger than S. Since $\varphi_r \in A(G_r) \subseteq \subseteq A_p(G_r)$, there exists $h_r \in A(G)$ such that $\varphi_r = h_r|_{G_r}$ and $\|\varphi_r\|_{A(G_r)} = \|h_r\|_{A(G)}$.

Hence

$$\|h_r\|_{A_p(G)} \leq \|h_r\|_{A(G)} = \|\varphi_r\|_{A(G_r)} = 1. \tag{3.5}$$

Let

$$g_r = \frac{a}{2i} \sum_{s=1}^{N} (\varphi_r^{\lambda_s} - \varphi_r^{-\lambda_s}).$$

Denote by f_r the real-valued function

$$f_r = \frac{a}{2i} \sum_{s=1}^{N} (h_r^{\lambda_s} - \bar{h}_r^{\lambda_s}).$$

Then $f_r|_{G_r} = g_r$ and, by (3.1) and (3.5),

$$\|f_r\|_{A_p(G)} \leq \|f_r\|_{A(G)} \leq r.$$

We have $e^{\frac{a}{2}(\varphi_r^{\lambda_s} - \varphi_r^{-\lambda_s})} = U_S + V_S$, where

$$U_S = \sum_{|n|<n_0} j_n \varphi_r^{n\lambda_s} \tag{3.6}$$

$$V_S = \sum_{|n|\geq n_0} j_n \varphi_r^{n\lambda_s}. \tag{3.7}$$

For every subset A of $\{1, 2, \cdots, N\}$ let us consider the product $\prod_{s\in A} U_S$; its Fourier coefficients are, by (3.3) and (3.4), products of the form $\prod_{s\in A} j_{n_s}$; consequently, if $|A|$ denotes the cardinality of A, one gets

$$\left\| \prod_{s\in A} U_S \right\|_{VN(G_r)} \leq j_0^{|A|}.$$

Since $e^{ig_r} = \prod_{l=1}^{N} (U_l + V_l)$, it follows

$$\|e^{ig_r}\|_{VN(G_r)} \leq \sum_A \left\| \prod_{s\in A} U_s \cdot \prod_{s\notin A} V_s \right\|_{VN(G_r)} \leq \sum_A \left\| \prod_{s\in A} U_s \right\|_{VN(G_r)} \cdot \prod_{s\notin A} \|V_s\|_{A(G_r)} \leq$$

$$\leq \sum_{l=1}^{N} \binom{N}{l} \left(\frac{1-J_0}{2}\right)^{N-l} (J_0)^l = \left(\frac{1+J_0}{2}\right)^N.$$

Therefore, by lemma 2

$$\| e^{if_r|_{G_r}} \|_{Cv_p(G_r)} \leq \left(\frac{1+j_0}{2}\right)^{N\sigma}.$$

Thus lemma 1 applies with $b = \sigma \log \left(\frac{2}{1+j_0}\right)$.

2) G has bounded exponent. As proved in [16], proposition 4, there exists a prime number q such that the continuous homomorphic images of G contain abelian subgroups of order q^α, with α arbitrarily large.

Therefore by Lemma 3, if $r \geq 1$ and $N = [r] + 1$, there exists an abelian group H_r, contained in some homomorphic image of G, with N independent characters. We denote by $\varphi_1, \cdots, \varphi_N$, these characters. They can be viewed as coordinate functions on the preimage G_r of H_r in G (see [16]). Let $h_s (s = 1, \cdots, N)$ be the extension preserving the $A(G_r)$-norm (as in the foregoing case) of φ_s to the whole of G. We shall distinguish two subcases.

a) $q \geq 3$.

By (3.1), (3.2) and elementary properties of Bessel functions one gets the following inequalities;

$$c_q = \sum_{k=-\infty}^{+\infty} j_{kq} < \sum_{k=-\infty}^{+\infty} j_{2k} = 1 \tag{3.8}$$

$$c_q > \sum_{k \neq 0} j_k \, . \tag{3.9}$$

Let $g_r = \frac{a}{2i} \sum_{s=1}^{N} (\varphi_s - \overline{\varphi}_s)$ and $f_r = \frac{a}{2i} \sum_{s=1}^{N} (h_s - \overline{h}_s)$; as before f_r is real-valued and $\|f_r\|_{A_p(G_r)} \leq r$.

One gets:

$$e^{ig_r} = \prod_{s=1}^{N} \left(\sum_{n=-\infty}^{+\infty} j_n \varphi_s^n \right) = \prod_{s=1}^{N} \left(\sum_{l=0}^{q-1} \left(\varphi_s^l \sum_{k=-\infty}^{+\infty} j_{kq+l} \right) \right) \tag{3.10}$$

and by (3.9)

$$\left| \sum_{k=-\infty}^{+\infty} j_{kq+l} \right| \leq \sum_{k \neq 0} |j_k| < c_q \qquad (l = 1, \cdots, q-1). \tag{3.11}$$

Because of (3.10) and the independence of the φ_s, $\| e^{ig_r} \|_{VN(G_r)}$ is the maximum of the numbers

$$\left| \prod_{s=1}^{N} \left(\sum_{k=-\infty}^{+\infty} j_{kq+l_i} \right) \right|$$

where l_i is any integer between 0 and $q-1$. By (3.11)

$$\| e^{ig_r} \|_{VN(G_r)} \leq (c_q)^N .$$

Therefore the theorem follows, by (3.8), from Lemmas 1 and 2 with $b = -\sigma \log (c_q)$.

b) $q = 2$.

In this case the theorem follows by a straightforward computation. Let $g_r = \frac{a}{2} \sum_{s=1}^{N} (\varphi_s + \overline{\varphi}_s)$. Since $\varphi_s(x) = \pm 1$ for every $x \in G_r$, g_r may be written as $a \sum_{s=1}^{N} \varphi_s$, and it can be extended as before to a real-valued function f_r on

G with $A_p(G)$-norm not larger than r. Therefore

$$e^{ig_r} = \prod_{s=1}^{N} (\cos(a\varphi_s) + i \sin(a\varphi_s)) =$$

$$= \prod_{s=1}^{N} (\cos a + i\, \varphi_s \sin a)\,.$$

By independence:

$$\| e^{ig_r} \|_{VN(G_r)} \leq \max\left((\cos a)^N \quad , \quad (\sin a)^N\right).$$

Let $c = \max(\cos a\,, \sin a)$; then, by Lemma 2

$$\| e^{if_r} \|_{CV_p(G)} \leq (c)^{N\sigma}\,.$$

By applying again Lemma 1, with $b = -\sigma \log(c)$, the theorem is completely proved.

References

[1] DE MICHELE L. AND SOARDI P. M. – *Functions which operate in the Fourier algebra of a discrete group*, « Proc. Amer. Math. Soc. », to appear.

[2] DRURY S. W. (1969) – *Studies in regular algebras*, Doctoral dissertation, Cambridge University.

[3] DUNKL C. F. (1969) – *Functions that operate in the Fourier algebra of a compact group*, « Proc. Amer. Math. Soc. », *21*, 540–544.

[4] EYMARD P. (1964) – *L'algèbre de Fourier d'un groupe localement compact*, « Bull. Soc. Math. France », *92*, 181–236.

[5] EYMARD P. (1969) – *Algèbres A_p et convoluteurs de L^p*, Séminaire Bourbaki, n. 367.

[6] FIGÀ-TALAMANCA A. (1964) – *Multipliers of p-integrable functions*, « Bull. Amer. Math. Soc. », *70*, 666–669.

[7] FIGÀ-TALAMANCA A. (1965) – *Translations invariant operators in L^p*, « Duke Math. J. », 32, 495–501.

[8] FISHER M. J. – *Multipliers and p-Fourier algebras*, submitted to Studia Math.

[9] HELSON H., KAHANE J. P. KATZNELSON Y. and RUDIN W. (1959) – *The functions which operate on Fourier transforms*, « Acta Math. », *102*, 135–157.

[10] HERZ C. S. (1965) – *Remarques sur la Note précédente de M. Varopoulos*, « C. R. Acad. Sci. Paris », *260*, 6001–6004.

[11] HERZ C. S. (1970) – *Le rapport entre l'algèbre A_p d'un groupe et d'un sous-groupe*, « C. R. Acad. Sci. Paris », *271*, 491–493.

[12] HERZ C. S. (1971) – *Une géneralization de la notion de transformée de Fourier Stieltjes*, McGill University, Montréal.

[13] HERZ C. S. (1971) – *The theory of p-spaces with an application to the convolution operators*, « Trans. Amer. Math. Soc. », *154*, 69–82.

[14] HUPPERT B. (1967) – *Endliche gruppen*, Springer Verlag, Berlin.

[15] MALLIAVIN M. P. AND MALLIAVIN P. (1972) – *Calcul symbolique sur l'algèbre de Wiener centrale du* SU3, Conference on harmonic analysis, lecture notes n. 266, Springer Verlag, Berlin.

[16] RIDER D. (1971) – *Functions which operate in the Fourier algebra of compact group*, « Proc. Amer. Math. Soc. », *28*, 525–530.

[17] RUDIN W. (1962) – *Fourier analysis on groups*, Interscience, New York.

[18] WATSON G. N. (1922) – *A treatise on the theory of Bessel functions*, Cambridge.