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A Goursat problem for a system of interacting waves

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Analisi matematica. — *A Goursat problem for a system of interacting waves* (*). Nota (**) di MEHMET NAMIK OĞUZTÖRELI, presentata dal Socio M. PICONE.

RIASSUNTO. — In questa Nota viene studiato un problema di Goursat per un sistema di equazioni di tipo iperbolico alle derivate parziali e alle differenze finite che descrive un sistema di onde in interazione fra prossimi vicini.

1. In the present paper we investigate a physical system S which consists of $n+1$ cascaded interacting subsystems S_k described by the equations

$$(1.1) \qquad S_k : \partial u_k = \gamma_k u_{k+1} - (\gamma_k + \gamma_{k-1}) u_k + \gamma_{k-1} u_{k-1} + v_k \qquad (k = 0, 1, \cdots, n)$$

subject to the conditions

$$(1.2) \qquad u_k \big|_{x=0} = u_k \big|_{y=0} = 0 \qquad (k = 0, 1, \cdots, n)$$

where γ_k's are certain constants, $v_k = v_k(x, y)$'s are certain given functions which are supposed to be continuous for all finite (x, y), $u_k = u_k(x, y)$'s are the unknown functions to be determined, and ∂ is the "total derivative" operator in the sense of M. Picone [1]:

$$(1.3) \qquad \partial = \frac{\partial^2}{\partial x \, \partial y} .$$

We assume that $\gamma_{-1} = \gamma_n = 0$ and $u_{-1} = u_{n+1} \equiv 0$.

Clearly, Eqs (1.3) represent a system of damped waves which are interacting according to the nearest neighbors actions. Let us note that when u_k and v_k's are functions of only one variable, say t, and the operator ∂ reduces to the ordinary differentiation with respect to t, $\partial = \frac{d}{dt}$, then the system (1.1)–(1.3) become the differential-difference system considered in [2] which occur in certain birth-and-death processes [3], in particle waves and deformation in crystalline solids [4], and in several mechanical problems and in the integration of certain partial differential equations [5].

Although the Goursat problem (1.1)–(1.3) can be reduced to the problem considered in [2] by the use of the techniques of Fourier transforms, we shall establish the solution directly by means certain forward and backward recurrence relations, employing the method used in [2].

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2. It is well known that the solution of the Goursat problem

$$\partial u + cu = f(x, y) \quad , \quad u\big|_{x=0} = u\big|_{y=0} = 0 \tag{2.1}$$

is given by the formula (cf. [7])

$$u(x, y) = \int_0^x d\xi \int_0^y R(x, y; \xi, \eta) f(\xi, \eta)\, d\eta \tag{2.2}$$

$R(x, y; \xi, \eta)$ being the Riemann function for the equation $\partial u + cu = 0$ which is of the form

$$R(x, y; \xi, \eta) = J(-c(x-\xi)(y-\eta)) , \tag{2.3}$$

where

$$J(t) = \sum_{j=0}^{\infty} \frac{t^j}{(j!)^2} . \tag{2.4}$$

(Compare also with [8]).

We now put

$$c_k = \gamma_k + \gamma_{k-1} \tag{2.5}$$

and

$$R_k(x, y; \xi, \eta) = J(-c_k(x-\xi)(y-\eta)). \tag{2.6}$$

Hence, the solution of the Goursat problem (1.1)–(1.3) is of the form

$$u_k(x, y) = V_k(x, y) + \gamma_k \int_0^x d\xi \int_0^y R_k(x, y; \xi, \eta)\, u_{k+1}(\xi, \eta)\, d\eta + \gamma_{k-1} \int_0^x d\xi \int_0^y R_k(x, y; \xi, \eta)\, u_{k-1}(\xi, \eta)\, d\eta , \tag{2.7}$$

where

$$V_k(x, y) = \int_0^x d\xi \int_0^y R_k(x, y; \xi, \eta)\, v_k(\xi, \eta)\, d\eta, \tag{2.8}$$

for $k = 0, 1, \cdots, n$. Clearly, all the functions $V_k(x, y)$ are completely determined by the given functions $v_k(x, y)$.

Since $u_{-1}(x, y) \equiv 0$, we have

$$u_0(x, y) = V_0(x, y) + \gamma_0 \int_0^x d\xi \int_0^y R_0(x, y; \xi, \eta)\, u_1(\xi, \eta)\, d\eta . \tag{2.9}$$

And since

$$(2.10) \qquad u_1(x,y) = V_1(x,y) + \gamma_1 \int_0^x d\xi \int_0^y R_1(x,y;\xi,\eta)\, u_2(\xi,\eta)\, d\eta$$
$$+ \gamma_0 \int_0^x d\sigma \int_0^y R_1(x,y;\sigma,\tau)\, u_0(\sigma,\tau)\, d\tau ,$$

we may write

$$(2.11) \qquad u_1(x,y) = \left\{ W_1(x,y) + \gamma_1 \int_0^x d\xi \int_0^y R_1(x,y;\xi,\eta)\, u_2(\xi,\eta)\, d\eta \right\}$$
$$+ \gamma_1^2 \int_0^x d\xi \int_0^y S_1(x,y,\xi,\eta)\, u_1(\xi,\eta)\, d\eta$$

by virtue of Eq (2.9), where

$$(2.12) \qquad W_1(x,y) = V_1(x,y) + \gamma_0 \int_0^x d\xi \int_0^y R_1(x,y;\xi,\eta)\, V_0(\xi,\eta)\, d\eta$$

and

$$(2.13) \qquad S_1(x,y;\xi,\eta) = \int_\xi^x d\sigma \int_\eta^y R_1(x,y;\sigma,\tau)\, R_0(\sigma,\tau;\xi,\eta)\, d\tau .$$

Since Eq (2.11) is a Volterra integral equation of the second kind with respect to $u_1(x,y)$, we have

$$(2.14) \qquad u_1(x,y) = Z_1(x,y) + \gamma_1 \int_0^x d\xi \int_0^y T_1(x,y;\xi,\eta)\, u_2(\xi,\eta)\, d\eta ,$$

where

$$(2.15) \qquad Z_1(x,y) = W_1(x,y) + \gamma_0^2 \int_0^x d\xi \int_0^y \Gamma_1(x,y;\xi,\eta;\gamma_0^2)\, W_1(\xi,\eta)\, d\eta$$

and

$$(2.16) \qquad T_1(x,y;\xi,\eta) = R_1(x,y;\xi,\eta) + \gamma_0^2 \int_\xi^x d\sigma \int_\eta^y \Gamma_1(x,y;\sigma,\tau)\, R_1(\sigma,\tau;\xi,\eta)\, d\tau$$

and $\Gamma_1(x,y;\xi,\eta;\lambda)$ is the resolvent of the kernel $S_1(x,y;\xi,\eta)$.

Now, putting

$$(2.17) \qquad Z_0(x,y) = V_0(x,y) , \; T_0(x,y;\xi,\eta) = R_0(x,y;\xi,\eta),$$

we assume that the following formula is valid for some k:

$$(2.18) \qquad u_{k-1}(x, y) = Z_{k-1}(x, y) + \gamma_{k-1} \int_0^x d\xi \int_0^y T_{k-1}(x, y; \xi, \eta)\, u_k(\xi, \eta)\, d\eta .$$

Then, by virtue of the Eqs (2.7) and (2.18), we may write

$$(2.19) \qquad u_k(x, y) = \{ W_k(x, y) + \gamma_k \int_0^x d\xi \int_0^y R_k(x, y; \xi, \eta)\, u_{k+1}(\xi, \eta)\, d\eta \} + \gamma_{k-1}^2 \int_0^x d\xi \int_0^y S_k(x, y; \xi, \eta)\, u_k(\xi, \eta)\, d\eta ,$$

where

$$(2.20) \qquad W_k(x, y) = V_k(x, y) + \gamma_{k-1} \int_0^x d\xi \int_0^y R_k(x, y; \xi, \eta)\, Z_{k-1}(\xi, \eta)\, d\eta,$$

and

$$(2.21) \qquad S_k(x, y; \xi, \eta) = \int_\xi^x d\sigma \int_\eta^y R_k(x, y; \sigma, \tau)\, T_{k-1}(\sigma, \tau; \xi, \eta)\, d\tau .$$

Since Eq (2.19) is a Volterra integral equation of the second kind in $u_k(x, y)$, we have

$$(2.22) \qquad u_k(x, y) = Z_k(x, y) + \gamma_k \int_0^x d\xi \int_0^y T_k(x, y; \xi, \eta)\, u_{k+1}(\xi, \eta)\, d\eta ,$$

where

$$(2.23) \qquad Z_k(x, y) = W_k(x, y) + \gamma_{k-1}^2 \int_0^x d\xi \int_0^y \Gamma_k(x, y; \xi, \eta; \gamma_{k-1}^2)\, W_k(\xi, \eta)\, d\eta$$

and

$$(2.24) \qquad T_k(x, y; \xi, \eta) = R_k(x, y; \xi, \eta) + \gamma_{k-1}^2 \int_\xi^x d\sigma \int_\eta^y \Gamma_k(x, y; \xi, \eta; \gamma_{k-1}^2)\, R_k(\sigma, \tau; \xi, \eta)\, d\tau,$$

and $\Gamma_k(x, y; \xi, \eta; \lambda)$ is the resolvent of the kernel $S_k(x, y; \xi, \eta)$. Thus the recurrence formula (2.18) is also valid for $k + 1$, if the functions W_k, Z_k, S_k and T_k are defined by the recurrence formulas (2.17), (2.20), (2.21), (2.23) and (2.24). Since formula (2.18) is true for $k = 1$ the induction is complete. Therefore formula (2.18) is valid for $k = 1, 2, \cdots, n-1$. Thus, we can write

$$u_{n-1}(x,y) = Z_{n-1}(x,y) + \gamma_{n-1} \int_0^x d\xi \int_0^y T_{n-1}(x,y;\xi,\eta)\, u_n(\xi,\eta)\, d\eta. \tag{2.25}$$

On the other hand, since $u_{n+1} \equiv 0$, we have

$$u_n(x,y) = V_n(x,y) + \gamma_{n-1} \int_0^x d\xi \int_0^y R_n(x,y;\xi,\eta)\, u_{n-1}(\xi,\eta)\, d\eta \tag{2.26}$$

by virtue of Eq (2.7) with $k = n$. Substituting the expression for u_{n-1} into Eq (2.26), we obtain the following Volterra integral equation

$$u_n(x,y) = W_n(x,y) + \gamma_{n-1}^2 \int_0^x d\xi \int_0^y S_n(x,y;\xi,\eta)\, u_n(\xi,\eta)\, d\eta \tag{2.27}$$

whose solution is of the form

$$u_n(x,y) = W_n(x,y) + \gamma_{n-1}^2 \int_0^x d\xi \int_0^y \Gamma_n(x,y;\xi,\eta;\gamma_{n-1}^2)\, W_n(\xi,\eta)\, d\eta. \tag{2.28}$$

Thus $u_n(x,y)$ is completely determined by the functions $W_n(x,y)$ and $\Gamma_n(x,y;\xi,\eta;\gamma_{n-1}^2)$.

We now substitute $u_n(x,y)$ given by Eq (2.28) into Eq (2.25). We then obtain

$$u_{n-1}(x,y) = F_{n-1}(x,y) + \gamma_{n-1} \int_0^x d\xi \int_0^y \Lambda_{n-1}(x,y;\xi,\eta)\, F_n(\xi,\eta)\, d\eta \tag{2.29}$$

where

$$\begin{cases} F_n(x,y) = W_n(x,y), \\ F_{n-1}(x,y) = Z_{n-1}(x,y) + \gamma_{n-1} \displaystyle\int_0^x d\xi \int_0^y T_{n-1}(x,y;\xi,\eta)\, F_n(\xi,\eta)\, d\eta \end{cases} \tag{2.20}$$

and

$$\begin{cases} \Lambda_n(x,y;\xi,\eta) = \gamma_{n-1}^2\, \Gamma_n(x,y;\xi,\eta;\gamma_{n-1}^2), \\ \Lambda_{n-1}(x,y;\xi,\eta) = \displaystyle\int_\xi^x d\xi \int_\eta^y T_{n-1}(x,y;\sigma,\tau)\, \Lambda_n(\sigma,\tau;\xi,\eta)\, d\tau. \end{cases} \tag{2.31}$$

Hence the function $u_{n-1}(x,y)$ is also completely determined.

We now assume that $u_k(x, y)$ admits a representation of the form

$$(2.32)\qquad u_k(x, y) = F_k(x, y) + \gamma_k \int_0^x d\xi \int_0^y \Lambda_k(x, y; \xi, \eta)\, F_n(\xi, \eta)\, d\eta\,.$$

Then, by virtue of Eqs (2.22) and (2.32), we can write

$$(2.33)\qquad u_{k-1}(x, y) = F_{k-1}(x, y) + \gamma_{k-1} \int_0^x d\xi \int_0^y \Lambda_{k-1}(x, y; \xi, \eta)\, F_n(\xi, \eta)\, d\eta\,,$$

where

$$(2.30)\qquad F_{n-1}(x, y) = Z_{k-1}(x, y) + \gamma_{k-1} \int_0^x d\xi \int_0^y T_{k-1}(x, y; \xi, \eta)\, F_k(\xi, \eta)\, d\eta$$

and

$$(2.34)\qquad \Lambda_{k-1}(x, y; \xi, \eta) = \lambda_k \int_\xi^x d\sigma \int_\eta^y T_{k-1}(x, y; \sigma, \tau)\, \Lambda_k(\sigma, \tau; \xi, \eta)\, d\tau\,.$$

Since the formula (2.32) is true for $k = n$, the formula (2.32) is valid for $k = n, n-1, n-2, \cdots, 1, 0$, if the functions F_{k-1} and Λ_{k-1} are defined by the backward recurrence relations (2.30) and (2.34).

Thus $\{u_0(x, y), u_1(x, y), \cdots, u_n(x, y)\}$, where $u_k(x, y)$ is given by the formula (2.32), constitutes the solution of the Goursat problem (I.1)-(I.2).

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