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The entropy rate admissibility criterion in thermoelasticity

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SEZIONE II

(Fisica, chimica, geologia, paleontologia e mineralogia)

Termodinamica. — *The entropy rate admissibility criterion in thermoelasticity* (*). Nota (**) di CONSTANTINE M. DAFERMOS, presentata, dal Socio Straniero C. TRUESDELL.

RIASSUNTO. — Si caratterizza l'ammissibilità di processi termodinamici con onde d'urto per corpi termoelastici mediante due criteri, uno dipendente dalla viscosità, l'altro sull'entropia, e si dimostra l'equivalenza di questi due criteri.

1. INTRODUCTION

The initial value problem for the dynamical equations of elasticity and thermoelasticity is not well posed in the class of smooth motions [1] so that an existence theory in the large can be established only for thermodynamic processes with shock waves. On the other hand, in this class of processes there is no uniqueness of solutions. This observation was made long ago for the equations of gas dynamics but in this case the entropy inequality rules out all but one solution and thus resolves the difficulty. Unfortunately, as we shall see below, the entropy inequality is not sufficiently powerful to single out a unique solution for general thermoelasticity. In consequence the theory must be supplemented with additional constitutive assumptions in the form of shock admissibility criteria. One method for motivating reasonable admissibility criteria is to visualize thermoelasticity as an approximation of a more elaborate theory for which uniqueness of solutions holds. In this spirit, we demonstrate in Section 3 that by visualizing a thermoelastic material as the limit of a family of thermoviscoelastic materials of the Voigt type one obtains a shock admissibility criterion, the *viscosity criterion*. An analogous idea has already been tested in the study of quasilinear hyperbolic systems and the experience accumulated so far seems to justify the approach (1).

Successful as the result may be, the approach is not entirely satisfactory. Indeed, conceptually it would be preferable to characterize admissibility of solutions by the internal structure of thermoelasticity theory itself rather than through an artificial extension. Somewhat surprisingly it turns out that the

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(1) For an elementary survey and a list of references see [2]. Although no general definitive results are yet available, it is known, for example, that the viscosity criterion guarantees existence, uniqueness and stability of solutions for the model equation $u_t + f(u)_X = 0$.

viscosity criterion is equivalent to a strengthened version of the second law of thermodynamics. Specifically we show that the solution which satisfies the viscosity criterion maximizes at each instant the rate of increase of the entropy of the body. Unfortunately the proof is computational and does not provide any clues on whether maximization of the entropy rate is a particular property of thermoelasticity or a general thermodynamic principle applicable to a broader class of materials.

In order to avoid cumbersome computations we discuss here admissibility of solutions for one-dimensional bodies only. The existence theory for quasi-linear hyperbolic systems suggests that the admissibility criteria should be developed in the class of functions of bounded variation. However, in order to avoid technical measure-theoretic arguments and without altering the spirit of the computations we will work here in the class of continuous and piecewise smooth thermodynamic processes with a finite number of shock waves. The proofs can be easily extended to the class of functions of bounded variation by using the apparatus developed by Volpert [3].

2. Thermoelasticity and Thermoviscoelasticity

We collect below the balance laws of continuum thermodynamics for one-dimensional bodies and the constitutive equations of thermoelastic and thermoviscoelastic materials. For a detailed systematic treatment the reader is referred to the articles of the Encyclopedia of Physics on Classical and Non-linear Field Theories [4, 5].

A thermodynamic process is the pair $(x(\mathrm{X}, t), \theta(\mathrm{X}, t))$, namely a motion and a temperature field. We shall use the notation $v = x_t$, $u = x_{\mathrm{X}}$, $d = x_{\mathrm{X}t}$, $g = \theta_{\mathrm{X}}$. In the class of continuous and piecewise smooth processes with shock waves, the balance laws and the principle of irreversibility, which are postulated in integral form, reduce to the field equations

$$\begin{aligned} \rho v_t &= \sigma_{\mathrm{X}} + \rho b \\ u_t &= v_{\mathrm{X}} = d \\ \rho\varepsilon_t &= \sigma d + h_{\mathrm{X}} + \rho r \end{aligned} \tag{2.1}$$

$$\rho\eta_t \geq \left(\frac{h}{\theta}\right)_{\mathrm{X}} + \rho\,\frac{r}{\theta} \tag{2.2}$$

at points of smoothness, and the jump conditions

$$\begin{aligned} &\rho\mathrm{U}\,[v] + [\sigma] = 0 \\ &\mathrm{U}\,[u] + [v] = 0 \\ &\rho\mathrm{U}\left[\varepsilon + \frac{1}{2}v^2\right] + [\sigma v + h] = 0 \end{aligned} \tag{2.3}$$

$$\rho\mathrm{U}\,[\eta] + \left[\frac{h}{\theta}\right] \leq 0 \tag{2.4}$$

across singular surfaces.

The set of balance laws is supplemented by constitutive equations for σ, η, h and the Helmholtz free energy $\psi = \varepsilon - \theta\eta$ which are required to satisfy the Clausius-Duhem-Truesdell and Toupin inequality (2.2) for all smooth thermodynamic processes which satisfy the balance laws (2.1). For thermoelasticity (where the independent variables are u, θ, g) this requirement yields constitutive equations of the form

$$(2.5) \qquad \psi = \hat{\psi}(u, \theta) \quad , \quad \sigma = \hat{\sigma}(u, \theta) \quad , \quad \eta = \hat{\eta}(u, \theta) \quad , \quad h = \hat{h}(u, \theta, g)$$

$$\hat{\sigma} = \rho \frac{\partial\hat{\psi}}{\partial u} \quad , \quad \hat{\eta} = -\frac{\partial\hat{\psi}}{\partial\theta}$$

$$\hat{h}(u, \theta, g)\, g \geq 0.$$

For thermoviscoelasticity (where the independent variables are u, d, θ, g) the constitutive equations are of the form

$$(2.6) \qquad \psi = \tilde{\psi}(u, \theta) \quad , \quad \sigma = \tilde{\sigma}(u, d, \theta, g) \quad , \quad \eta = \tilde{\eta}(u, \theta) \quad , \quad h = \tilde{h}(u, d, \theta, g)$$

$$\tilde{\sigma} = \rho \frac{\partial\tilde{\psi}}{\partial u} + \tilde{\tau}(u, d, \theta, g) \quad , \quad \tilde{\eta} = -\frac{\partial\tilde{\psi}}{\partial\theta}$$

$$\tilde{\tau}(u, d, \theta, g)\, d + \frac{\tilde{h}(u, d, \theta, g)}{\theta}\, g \geq 0.$$

3. THE VISCOSITY CRITERION

Consider a thermoelastic body $\mathcal{M}$ characterized by its constitutive equations $\hat{\psi}, \hat{\sigma}, \hat{\eta}, \hat{h}$ of the form (2.5). We visualize $\mathcal{M}$ as a limiting member of a family of thermoviscoelastic bodies which is constructed by the following procedure: We begin with functions $\tilde{\tau}(u, d, \theta, g)$ and $\tilde{h}(u, d, \theta, g)$ which satisfy $(2.6)_3$ and for each $\mu > 0$ we consider the body $\mathcal{M}_\mu$ with constitutive equations

$$(3.1) \qquad \tilde{\psi}_\mu(u, \theta) = \hat{\psi}(u, \theta) \quad , \quad \tilde{\sigma}_\mu(u, d, \theta, g) = \hat{\sigma}(u, \theta) + \tilde{\tau}(u, \mu d, \theta, \mu g)$$

$$\tilde{\eta}_\mu(u, \theta) = \hat{\eta}(u, \theta) \quad , \quad \tilde{h}_\mu(u, d, \theta, g) = \hat{h}(u, \theta, g) + \tilde{h}(u, \mu d, \theta, \mu g).$$

Observing that $(2.6)_3$ implies, in particular, $\tilde{\tau}(u, 0, \theta, 0) = 0$, $\tilde{h}(u, 0, \theta, 0) = 0$, we conclude that the constitutive equations of $\mathcal{M}_\mu$ reduce as $\mu \to 0^+$ to the constitutive equations of $\mathcal{M}$. This observation motivates the following admissibility criterion for solutions: We shall say that a continuous and piecewise smooth thermodynamic process $(x(X, t), \theta(X, t))$, satisfying the balance laws, is admissible if for each $\mu > 0$ there is a smooth process $(x_\mu(X, t), \theta_\mu(X, t))$ of $\mathcal{M}_\mu$ which satisfies the balance laws and tends to $(x(X, t), \theta(X, t))$ as $\mu \to 0^+$. Since the viscosity criterion is not the main object of this note we shall not make the sense of convergence precise but, roughly speaking, we require that for small μ $(x_\mu(X, t), \theta_\mu(X, t))$ approximates the profiles of the shock waves of $(x(X, t), \theta(X, t))$.

An important feature of the above criterion is that it is independent of the choice of $\dot{\tilde{\tau}}$, $\tilde{h}$ and can be expressed as an admissibility condition on the shock waves of $(x(X, t), \theta(X, t))$ in terms of the constitutive equations of $\mathcal{M}$ itself. This fact is familiar from the theory of hyperbolic systems (see, e.g., [6] for a general description of the idea and [7] for a systematic discussion) so that we shall only present here a brief sketch of the derivation of the shock admissibility condition.

Let $(\bar{X}, \bar{t})$ be a fixed point on a shock wave of $(x(X, t), \theta(X, t))$ and let U be the speed of propagation and $(u_-, v_-, \bar{\theta}, g_-)$, $(u_+, v_+, \bar{\theta}, g_+)$ be the limits from the left and right. For $\mu > 0$ we introduce new coordinates $\zeta = \frac{1}{\mu}\{U(X - \bar{X}_\mu) + t - \bar{t}_\mu\}$, $\xi = \frac{1}{\mu}\{X - \bar{X}_\mu - U(t - \bar{t}_\mu)\}$, tangential and normal to the shock wave, where $(\bar{X}_\mu, \bar{t}_\mu)$ tends to $(\bar{X}, \bar{t})$ as $\mu \to 0^+$. We write $(x_\mu(X, t), \theta_\mu(X, t))$ and (2.1) in terms of ζ, ξ and we let $\mu \to 0^+$. Since $(x_\mu(X, t), \theta_\mu(X, t))$ approximates the profiles of the shock waves of $(x(X, t), \theta(X, t))$, (u_μ, v_μ, g_μ) must tend, as $\mu \to 0^+$, to functions (u_0, v_0, g_0) of ξ alone which satisfy the system of ordinary differential equations

$$
\begin{aligned}
(3.2)\qquad & \rho U \dot{v}_0(\xi) + \dot{\hat{\sigma}}(u_0(\xi), \bar{\theta}) + \dot{\tilde{\tau}}(u_0(\xi), \dot{v}_0(\xi), \bar{\theta}, 0) = 0 \\
& U \dot{u}_0(\xi) + \dot{v}_0(\xi) = 0 \\
& \rho U \dot{\hat{\varepsilon}}(u_0(\xi), \bar{\theta}) + (\hat{\sigma}(u_0(\xi), \bar{\theta}) + \dot{\tilde{\tau}}(u_0(\xi), \dot{v}_0(\xi), \bar{\theta}, 0))\, \dot{v}_0(\xi) \\
& \qquad + \hat{h}(u_0(\xi), \bar{\theta}, g_0(\xi)) + \tilde{h}(u_0(\xi), \dot{v}_0(\xi), \bar{\theta}, 0) = 0 .
\end{aligned}
$$

Eliminating $\dot{v}_0(\xi)$ between $(3.2)_1$ and $(3.2)_2$ and integrating the resulting equation once we conclude that for every point $\bar{u}$ between u_- and u_+ the interval defined by $\bar{u}$ and u_+ is positive invariant under the dynamical system generated by the equation

$$
\begin{aligned}
(3.3)\qquad \dot{\tilde{\tau}}(u_0(\xi), -U\dot{u}_0(\xi), \bar{\theta}, 0) &= \rho U^2 (u_0(\xi) - u_-) \\
&\quad - (\hat{\sigma}(u_0(\xi), \bar{\theta}) - \hat{\sigma}(u_-, \bar{\theta})) .
\end{aligned}
$$

Observing that $(2.6)_3$ implies, in particular, $\dot{\tilde{\tau}}(u, d, \theta, 0)\, d \geq 0$, we deduce that the above condition is equivalent to

$$
(3.4)\qquad \frac{\hat{\sigma}(\bar{u}, \bar{\theta}) - \hat{\sigma}(u_-, \bar{\theta})}{\bar{u} - u_-} - \rho U^2 \begin{cases} \geq 0 & \text{if } U > 0 \\ \leq 0 & \text{if } U < 0 \end{cases}
$$

for every $\bar{u}$ between u_- and u_+. Noting that on account of (2.3) $\rho U^2 = [\sigma][u]^{-1}$, we conclude that (3.4) is equivalent to the following statement: if $(u_+ - u_-)U > 0$ (or $(u_+ - u_-)U < 0$) the chord which joins $(u_-, \hat{\sigma}(u_-, \bar{\theta}))$ with $(u_+, \hat{\sigma}(u_+, \bar{\theta}))$ lies below (or above) the graph of $\hat{\sigma}(\cdot, \bar{\theta})$ between u_- and u_+ (compare with Wendroff [8]). This condition will be called the *viscosity admissibility criterion*.

We now study the relationship between the viscosity criterion and the entropy inequality (2.4). Eliminating $[h]$ between $(2.3)_3$ and (2.4) and using $(2.3)_1$ and $(2.3)_2$ we arrive at

$$
\begin{aligned}
(3.5) \qquad \rho U \bar{\theta}\, [\eta] + [h] &= -\rho U \left[\psi + \frac{1}{2} v^2\right] - [\sigma v] \\
&= U \left\{ \frac{1}{2} (\sigma_+ + \sigma_-)\, [u] - \rho\, [\psi] \right\} \\
&= U \left\{ \frac{1}{2} (\hat{\sigma}(u_+ , \bar{\theta}) + \hat{\sigma}(u_- , \bar{\theta}))\, (u_+ - u_-) - \int_{u_-}^{u_+} \hat{\sigma}(u , \bar{\theta})\, du \right\}.
\end{aligned}
$$

We observe that the right-hand side of (3.5) is U times the (signed) area between the chord which joins $(u_- , \hat{\sigma}(u_- , \bar{\theta}))$ with $(u_+ , \hat{\sigma}(u_+ , \bar{\theta}))$ and the graph of $\hat{\sigma}(\cdot , \bar{\theta})$ between u_- and u_+. Therefore, every solution which satisfies the viscosity criterion satisfies automatically the entropy inequality (2.4). The converse, however, is not generally true. Indeed, (2.4) and the viscosity criterion are equivalent only when for each fixed $\bar{\theta}$ $\hat{\sigma}(\cdot , \bar{\theta})$ is convex or concave. This explains why for polytropic gases the entropy inequality is sufficient to single out the admissible solution.

4. THE ENTROPY RATE ADMISSIBILITY CRITERION

In the previous section we have seen that the entropy inequality is in general weaker than the viscosity criterion. Here we show that the viscosity criterion follows from a strengthened version of the second law af thermodynamics.

DEFINITION. *A continuous and piecewise smooth thermodynamic process* $(x(X , t) , \theta(X , t))$, *satisfying the balance laws, is admissible according to the entropy rate criterion if for each* $\tau \geq 0$ *it maximizes the rate of increase of the entropy of the body over the set of processes which satisfy the balance laws and coincide with* $(x(X , t) , \theta(X , t))$ *for* $t \leq \tau$.

The following proposition establishes the equivalence of the viscosity and the entropy rate criterion.

THEOREM. *Every continuous and piecewise smooth thermodynamic process which satisfies the viscosity criterion satisfies also the entropy rate admissibility criterion.*

The proof proceeds as follows: We consider a thermoelastic body with reference configuration the interval $[a , b]$ and constitutive equations (2.5). For any continuous and piecewise smooth process $(x(X , t) , \theta(X , t))$ with a finite number of shock waves (2) which satisfies the balance laws, the rate

(2) For normalization we assume that each shock wave is defined on a time interval which is closed from below and open from above.

of entropy increase at $\tau \geq 0$ is given by

$$\frac{D^+}{Dt}\int_a^b \rho\eta\, dx = -\sum_{\text{shocks}} \rho U[\eta] + \sum \int \rho\eta_t\, dx \tag{4.1}$$

where the first summation runs over all shock waves which intersect $[a, b]$ at time τ and the second summation runs over all intervals into which $[a, b]$ is partitioned by the above shock waves. At points of smoothness one deduces with the help of (2.1) and (2.5)

$$\rho\eta_t = \left(\frac{h}{\theta}\right)_X + \frac{h\theta_x}{\theta^2} + \rho\frac{r}{\theta}$$

so that (4.1) gives

$$\frac{D^+}{Dt}\int_a^b \rho\eta\, dx = -\sum_{\text{shocks}} \left(\rho U[\eta] + \left[\frac{h}{\theta}\right]\right) + \int_a^b \left(\frac{h\theta_x}{\theta^2} + \rho\frac{r}{\theta}\right) dx + \frac{h}{\theta}\Big|_a^b . \tag{4.2}$$

Using (3.5) we rewrite (4.2) in the form

$$\frac{D^+}{Dt}\int_a^b \rho\eta\, dx = -\sum_{\text{shocks}} \frac{U}{\theta}\left\{\frac{1}{2}(\hat\sigma(u_+, \theta) + \hat\sigma(u_-, \theta))(u_+ - u_-) - \right. \tag{4.3}$$
$$\left. - \int_{u_-}^{u_+} \hat\sigma(u, \theta)\, du\right\} + \int_a^b \left(\frac{h\theta_x}{\theta^2} + \rho\frac{r}{\theta}\right) dx + \frac{h}{\theta}\Big|_a^b .$$

Thus, if two processes coincide for $t \leq \tau$, their rates of entropy increase at τ may differ at most in the summation term on the right hand side of (4.3) and this only at points where new shock waves are branching out. The comparison of such terms is quite tedious and is contained in the proof of Theorem 2 of [9] so that it will not be repeated here. The comparison verifies that the rate of entropy increase is maximized by the process which satisfies the viscosity criterion.

The theorem justifies the entropy rate criterion but the proof leaves something to be desired since it does not reveal any intrinsic reason why the proposition is true and thus does not provide any clues on whether maximization of the entropy rate is a particular property of thermoelasticity or a general thermodynamic principle.

It is conceivable that the solution which maximizes the rate of entropy increase will also maximize at each instant the entropy of the body over the class of processes which satisfy the balance laws and assigned initial and boundary conditions. However, I have been unable to verify this conjecture.

There are strong indications that the viscosity (and hence the entropy rate) criterion guarantees existence, uniqueness and stability of solutions but so far this has been established rigorously only for the model equation $u_t + f(u)_X = 0$.

It is possible to characterize admissible solutions of the equations of nonlinear hyperelasticity by an analogous criterion. The appropriate requirement is that admissible solutions minimize the rate of decrease of the mechanical energy of the body. The equivalence of this and the viscosity criterion has been established in [9].

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