Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Canonical systems on flag manifolds

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.1-2, p. 10–17.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_1-2_10_0>

Matematica. — *Canonical systems on flag manifolds* (*). Nota (**) di SAMUEL A. ILORI, presentata dal Socio B. SEGRE.

RIASSUNTO. — Vengono calcolate le classi di Chern delle varietà di bandiere di lunghezza 1 e costruiti geometricamente tutti i sistemi canonici di certe grassmanniane.

§ 1. INTRODUCTION

Given integers $q_1, \cdots, q_m$ such that

$$0 \leq q_1 \leq q_2 \leq \cdots \leq q_m = n,$$

one defines a $(q_1, \cdots, q_m)$–flag as a nested system

$$S: \quad S_{q_1} \subset S_{q_2} \subset \cdots \subset S_{q_m}, \qquad \dim \ S_{q_i} = q_i,$$

of subspaces of S_n, the complex n–dimensional projective space. The set of all such flags is called an incomplete flag manifold in S_n and is denoted by $W(q_1, \cdots, q_m)$, where $q_1, \cdots, q_m$ are called the flag–dimensions of $W(q_1, \cdots, q_m)$ (cfr. [1], [2]). Special cases are the complete flag manifold, $W(0, \cdots, n)$, which is denoted by $F(n+1)$, the flag manifold of length r, $W(0, \cdots, r, n)$ and the Grassnannian, $W(m, n)$. In § 2, we shall find explicitly all the Chern classes of $W(0, 1, n)$. In § 3, using the Ehresmann subvarieties introduced in [2], we shall also construct geometrically all the canonical systems of $W(1, 6)$.

§ 2. CHERN CLASSES

A basis for $H^*(W(0, \cdots, r, n))$ is a set of monomials

$$\gamma_0^{a_0} \gamma_1^{a_1} \cdots \gamma_r^{a_r}$$

where the a_i are integers subject to the restriction

$$0 \leq a_i \leq n - i, \qquad (i = 0, \cdots, r)$$

(cfr. Theorem 1, p. 4–19 in [5]). In this section, we find the various Chern classes of $W(0, 1, n)$ in terms of the above basis. First we find $H^*(W(0, 1, n))$ in two ways with two sets of basis elements and find a relationship between them.

(*) *AMS(MOS) subject classifications*: Primary 14M15, 55F40 and 57D20; *Key words*. Ehresmann subvarieties, $W(0, \cdots, r, n)$, $W(m, n)$.

(**) Pervenuta all'Accademia l'8 agosto 1974.

PROPOSITION 2.1. *The cohomology ring of* W (0, 1, n) *is given either as*

$$H^* (W(0, 1, n)) = Z[\delta, \eta]$$

subject to

$$\delta^{n+1} = 0 \quad \text{and} \quad \eta^n = -\sum_{i=1}^{n} \binom{n+1}{i} \delta^i \eta^{n-i},$$

or as

$$H^* (W(0, 1, n)) = Z[\gamma_0, \gamma_1]$$

subject to,

$$\bar{\sigma}_n (\gamma_0, \gamma_1) = 0 = \bar{\sigma}_{n+1} (\gamma_0, \gamma_1).$$

Moreover, the two sets of generators are related in the following way:

$$\delta = \gamma_1 \quad , \quad \eta = \gamma_0 - \gamma_1.$$

Proof. W (0, 1, n) can be considered as a bundle

$$W(0, 1, n) \xrightarrow[\pi]{} W(0, n), \qquad \text{fibre} \quad W(0, n-1).$$

It, therefore, follows from Theorem 1, p. 4–19 in [5] that

$$H^* (W(0, 1, n)) = \pi^* H^* (W(0, n)) [\eta]$$

subject to

$$\sum_{i=0}^{n} (-1)^i \pi^* (c_i (W(0, n)) \eta^{n-i} = 0.$$

But $H^* (W(0, n)) = Z[x]$ subject to $x^{n+1} = 0$ and the first part of the Proposition follows by substituting for $c_i (W(0, n))$ from p. 70 of [3] and putting $\delta = \pi^* x$. When W (0, 1, n) is considered as a flag manifold, then from Theorem 1, p. 4–19 in [5], we have that

$$H^* (W(0, 1, n)) = Z[\gamma_0, \gamma_1, \{\sigma_i (\gamma_2, \cdots, \gamma_n) \mid i = 1, \cdots, n-1\}]$$

subject to

$$\prod_{i=0}^{n} (1 - \gamma_i) = 1,$$

from which the second part follows easily. To find a relationship among the generators, we first put

$$\gamma_0 = \eta + \delta \quad , \quad \gamma_1 = \delta$$

and assume that

$$\bar{\sigma}_n (\eta + \delta, \delta) = 0 = \bar{\sigma}_{n+1} (\eta + \delta, \delta).$$

Note that

$$(2.2)\qquad \bar{\sigma}_n(\eta+\delta\,,\delta)=\sum_{i=0}^{n}(\eta+\delta)^{n-i}\,\delta^i=\sum_{r=0}^{n}\left[\sum_{s=0}^{n-r}\binom{n-s}{r}\right]\eta^r\,\delta^{n-r}=$$

$$=\sum_{r=0}^{n}\binom{n+1}{r+1}\eta^r\,\delta^{n-r}.$$

Hence

$$\eta^n=-\sum_{i=1}^{n}\binom{n+1}{i}\delta^i\,\eta^{n-i}.$$

Also, since $\bar{\sigma}_n(\eta+\delta\,,\delta)=0=\bar{\sigma}_{n+1}(\eta+\delta\,,\delta)$, we have that

$$\bar{\sigma}_{n-1}(\eta+\delta\,,\delta)\cdot\sigma_2(\eta+\delta\,,\delta)=0.$$

But

$$\bar{\sigma}_{n-1}(\eta+\delta\,,\delta)\cdot\sigma_2(\eta+\delta\,,\delta)=(\eta+\delta)\,\delta\sum_{r=0}^{n-1}\binom{n}{r+1}\eta^r\,\delta^{n-1-r}=$$

$$=\eta^n\,\delta+\sum_{r=0}^{n-2}\binom{n}{r+1}\eta^{r+1}\,\delta^{n-r}+\sum_{r=1}^{n-1}\binom{n}{r+1}\eta^r\,\delta^{n+1-r}+n\delta^{n+1}=$$

$$=-\sum_{r=1}^{n}\binom{n+1}{r}\delta^{r+1}\,\eta^{n-r}+\sum_{r=1}^{n-1}\left[\binom{n}{r}+\binom{n}{r+1}\right]\eta^r\,\delta^{n+1-r}+n\delta^{n+1}=$$

$$=-(n+1)\,\delta^{n+1}-\sum_{r=1}^{n-1}\binom{n+1}{n-r}\delta^{n+1-r}\,\eta^r+\sum_{r=1}^{n-1}\binom{n+1}{r+1}\eta^r\,\delta^{n+1-r}+n\delta^{n+1}=$$

$$=-\,\delta^{n+1}.$$

Thus $\delta^{n+1}=0$.

Conversely, we put $\delta=\gamma_1\,,\eta=\gamma_0-\gamma_1$ and assume that

$$\gamma_1^{n+1}=0\qquad\text{and}\qquad(\gamma_0-\gamma_1)^n=-\sum_{i=1}^{n}\binom{n+1}{i}\gamma_1^i\,(\gamma_0-\gamma_1)^{n-i}.$$

$$\bar{\sigma}_n(\gamma_0\,,\gamma_1)=\sum_{r=0}^{n}\binom{n+1}{r+1}(\gamma_0-\gamma_1)^r\,\gamma_1^{n-r},\qquad\text{(by (2.2))},$$

$$=0\,,\qquad\text{(by hypothesis)},$$

and $\bar{\sigma}_{n+1}(\gamma_0\,,\gamma_1=\gamma_1^{n+1}+\gamma_0\,\bar{\sigma}_n(\gamma_0\,,\gamma_1)$

$$=0\,,\qquad\text{(by the above and the hypothesis)}.$$

PROPOSITION 2.3. *Let* V^* *be the tangent direction bundle of an algebraic variety* V *of complex dimension* n. *Then the total Chern class of* V *is given by*

$$c(V^*)=\pi^*\,c(V)\cdot\sum_{i=0}^{\infty}(1-\eta)^{n-i}\,\pi^*\,c_i(V)\,,$$

subject to

$$\sum_{i=0}^{n} (-1)^i \pi^* c_i(\mathrm{V}) \eta^{n-i} = 0,$$

where $\pi : \mathrm{V}^* \to \mathrm{V}$, *fibre* $\mathrm{W}(0, n-1)$. *Furthermore,*

$$c_h(\mathrm{V}^*) = \sum_{i=0}^{h} \sum_{s=0}^{n} (-1)^{h-i} \binom{n-s}{h-i} \pi^* (c_s(\mathrm{V})\, c_{i-s}(\mathrm{V}))\, \eta^{h-i}.$$

Proof. The result follows immediately from p. 41 of [4] and section 15 of [71].

COROLLARY 2.4. *Let* $\mathrm{W}(0, 1, n)$ *be a flag manifold of length* 1. *Then*

$$c_h(\mathrm{W}(0, 1, n)) = \sum_{i=\max(0, h-n)}^{\min(h,n)} \sum_{s=0}^{n} (-1)^h \binom{n-s}{h-i} \binom{n+1}{s} \binom{n+1}{i-s} \delta^i \eta^{h-i},$$

subject to

$$\eta^n = -\sum_{i=1}^{n} \binom{n+1}{i} \delta^i \eta^{n-i}.$$

Proof. $\mathrm{W}(0, 1, n)$ is a tangent direction bundle and hence from the Proposition, we have that

$$c_h(\mathrm{W}(0, 1, n)) =$$

$$= \sum_{i=0}^{h} \sum_{s=0}^{n} (-1)^{h-i} \binom{n-s}{h-i} (-1)^s \binom{n+1}{s} \delta^s (-1)^{i-s} \binom{n+1}{i-s} \delta^{i-s} \eta^{h-i}$$

(Cfr. p. 70 of [3]).

The result now follows since $\delta^{n+1} = 0$ and the highest power of η is η^n. The condition

$$\eta^n = -\sum_{i=1}^{n} \binom{n+1}{i} \delta^i \eta^{n-i}$$

also follows from the Proposition and p. 70 of [3].

§ 3. CANONICAL SYSTEMS ON W (1, 6).

LEMMA 3.1. *The elements*

$$\sigma_i(\gamma_0, \cdots, \gamma_m) \in \mathrm{H}^{2i}(\mathrm{W}(m, n)),$$

$1 \leq i \leq m+1$, *generate the Z–algebra* $\mathrm{H}^*(\mathrm{W}(m, n))$ *subject to*

$$\bar{\sigma}_{m+i}(\gamma_0, \cdots, \gamma_m) = 0, \qquad i \geq 0.$$

Proof. The proof follows from Theorem 1, p. 4–19 in [5].

LEMMA 3.2. *In the notation of* [2],

$$\sigma_h (n , q ; \mathrm{F}) = (-1)^h \cdot w_{h,1} (q ; \mathrm{F}), \qquad where \quad \mathrm{F} = \mathrm{F} (n + 1).$$

Proof. Consider the following commutative diagram of fibre bundles

$$\begin{array}{ccc} \mathrm{GL}(n+1) & \xrightarrow[\lambda]{\mathrm{GL}(q+1,n-q)} & \mathrm{W}(q,n) \\ & {\scriptstyle \Delta(n+1)} \searrow^{\xi} \quad \nearrow_{\pi} & \\ & \mathrm{F}(n+1) & \end{array}$$

Let λ', λ'' be the sub-bundle and quotient-bundle associated with the principal $\mathrm{GL}(q+1 , n-q)$–bundle λ over $\mathrm{W}(q , n)$. Thus λ' is an $(q+1)$–dimensional vector bundle over $\mathrm{W}(q , n)$ such that

$$c_h (\lambda') = \pi^* \sigma_h (\gamma_0 , \cdots , \gamma_q) , \qquad h = 1 , \cdots , q + 1,$$
$$= \sigma_h (n , q ; \mathrm{F}).$$

By considering the universal bundle over $\mathrm{W}(q , n)$, the h^{th}–characteristic class is the Schubert subvariety representing the $[q]$'s which meet a fixed $[n-q+h-2]$ in an $[h-1]$, i.e. $w_{h,1} (q ; \mathrm{F})$ (cfr. [6]). The result now follows since

$$w_{h,1} (q ; \mathrm{F}) = (-)^h c_h (\lambda') \qquad \text{(cfr. 29.4 of [7 II]).}$$

PROPOSITION 3.3. *The canonical systems of the Grassmannian* $\mathrm{W} = \mathrm{W}(1 , 6)$, *are given by* (*where we write* $[a , b]$ *for* $[(a , b) ; \mathrm{W}]$):

$$\mathrm{X}_0 (\mathrm{W}) = 21 [5 , 6] ,$$

$$\mathrm{X}_1 (\mathrm{W}) = - 105 [4 , 6] ,$$

$$\mathrm{X}_2 (\mathrm{W}) = 245 [3 , 6] + 140 [4 , 5] ,$$

$$\mathrm{X}_3 (\mathrm{W}) = - 315 [2 , 6] - 350 [3 , 5] ,$$

$$\mathrm{X}_4 (\mathrm{W}) = 217 [1 , 6] + 406 [2 , 5] + 252 [3 , 4] ,$$

$$\mathrm{X}_5 (\mathrm{W}) = - 63 [0 , 6] - 238 [1 , 5] - 336 [2 , 4] ,$$

$$\mathrm{X}_6 (\mathrm{W}) = 57 [0 , 5] + 191 [1 , 4] + 143 [2 , 3] ,$$

$$\mathrm{X}_7 (\mathrm{W}) = - 42 [0 , 4] - 98 [1 , 3] ,$$

$$\mathrm{X}_8 (\mathrm{W}) = 22 [0 , 3] + 25 [1 , 2] ,$$

$$\mathrm{X}_9 (\mathrm{W}) = - 7 [0 , 2] .$$

Proof. By the remark in section 1.4 of [2], the following table gives an additive basis of $H^*(W(1,6))$ both in terms of Ehresmann varieties and in terms of the γ_j, where we put $\sigma_j = \sigma_j(\gamma_0, \gamma_1)$, $\bar{\sigma}_j = \bar{\sigma}_j(\gamma_0, \gamma_1)$.

DIMENSION	EHRESMANN BASE	BASE IN TERMS OF THE γ_j
10	[0,1]	1
9	[0,2]	$-\sigma_1$
8	[0,3]	$\bar{\sigma}_2$
	[1,2]	σ_2
7	[0,4]	$-\bar{\sigma}_3$
	[1,3]	$-\sigma_1\sigma_2$
6	[0,5]	$\bar{\sigma}_4$
	[1,4]	$\sigma_2\bar{\sigma}_2$
	[2,3]	σ_2^2
5	[0,6]	$-\bar{\sigma}_5$
	[1,5]	$-\sigma_2\bar{\sigma}_3$
	[2,4]	$-\sigma_1\sigma_2^2$
4	[1,6]	$\sigma_2\bar{\sigma}_4$
	[2,5]	$\sigma_2^2\bar{\sigma}_2$
	[3,4]	σ_2^3
3	[2,6]	$-\sigma_2^2\bar{\sigma}_3$
	[3,5]	$-\sigma_1\sigma_2$
2	[3,6]	$\sigma_2^3\bar{\sigma}_2$
	[4,5]	σ_2^4
1	[4,6]	$-\sigma_1\sigma_2^4$
0	[5,6]	σ_2^5

From 29.4 of [7 II], the canonical systems on $W(1,6)$ are related to the Chern classes of $W(1,6)$ by the formula

$$X_{10-h}(W) = (-)^h c_h(W), \qquad h = 1, \cdots, 10.$$

From p. 522 of [7 I], the total Chern class of $W(1,6)$ is given by $c(W(1,6)) = \left[(1+\sigma_1)^7 + 7(1+\sigma_1)^6\sigma_2 + \binom{7}{2}(1+\sigma_1)^5\sigma_2^2 + \cdots + \sigma_2^7\right] \times$ $\times\, [1 + (\gamma_1 - \gamma_0)^2 + \cdots + (\gamma_1 - \gamma_0)^{10}]$.

Since negative generators have been used in the calculation of the total Chern class, it follows that

$$X_{10-h}(W) = c_h(W).$$

We now calculate the Chern classes of $W(1, 6)$ using the fact that from Lemma 3.1, $\overline{\sigma}_{6+i} = 0$, $i \geq 0$, the identities $\Sigma\,(-)^h\,\sigma_h\,\overline{\sigma}_{i-h} = 0$, Lemma 3.2 and Monk's Intersection formula [8]:

$$c_1 = -7\,\sigma_1 = -7\,[0, 2]^*,$$

$$c_2 = \left[\binom{7}{2} + 1\right]\sigma_1^2 = 3\,\sigma_2 = 22\,w(1)^2 + 3\,[1, 2]^* = 22\,[0,3]^* + 25\,[1, 2]^*,$$

$$\begin{aligned} c_3 &= \binom{7}{3}\sigma_1^3 + 42\,\sigma_1\,\sigma_2 + 7\,\sigma_1(\sigma_1^2 - 4\,\sigma_2) = 42\,\sigma_1^3 = 14\,\sigma_1\,\sigma_2 = -42\,w(1)^3 - \\ &- 14\,w(1)\,[1, 2]^* = -42\,[0, 4]^* - 98\,[1, 3]^*, \end{aligned}$$

$$\begin{aligned} c_4 &= \binom{7}{4}\sigma_1^4 + \binom{7}{2}\sigma_1^2(\sigma_1^2 - 4\,\sigma_2) + (\sigma_1^2 - 4\,\sigma_2)^2 + 7\,\sigma_2\left[\binom{6}{2}\sigma_1^2 + \sigma_1^2 - 4\,\sigma_2\right] + \\ &+ \binom{7}{2}\sigma_2^2 = 57\,\sigma_1^4 + 20\,\sigma_1^2\,\sigma_2 + 9\,\sigma_2^2 = 57\,\sigma_1^4 + 29\,\sigma_1^2\,\sigma_2 - 9\,\overline{\sigma}_3\,\sigma_1 + 9\,\overline{\sigma}_4 = \\ &= 57\,w(1)^4 + 29\,w(1)^2\,[1, 2]^* - 9\,w(1)\,[0, 4]^* + 9\,[0, 5]^* = 57\,[0, 5]^* + \\ &+ 19\,[1, 4]^* + 143\,[2, 3]^*, \end{aligned}$$

$$\begin{aligned} c_5 &= \binom{7}{5}\sigma_1^5 + \binom{7}{3}\sigma_1^3(\sigma_1^2 - 4\,\sigma_2) + \binom{7}{1}\sigma_1(\sigma_1^2 - 4\,\sigma_2)^2 + \\ &+ 7\,\sigma_2\left[\binom{6}{3}\sigma_1^3 + \binom{6}{1}\sigma_1(\sigma_1^2 - 4\,\sigma_2)\right] + \binom{7}{2}\sigma_2^2\binom{5}{1}\sigma_1 = \\ &= 63\,\sigma_1^5 + 35\,\sigma_1^3\sigma_2 - 49\,\sigma_1^2\,\overline{\sigma}_3 + 49\,\sigma_1\,\overline{\sigma}_4 = -63\,[0,6]^* - 238\,[1,5]^* - 336\,[2,4]^*, \end{aligned}$$

$$\begin{aligned} c_6 &= (\sigma_1^2 - 4\,\sigma_2)\,c_4 + \binom{7}{6}\sigma_1^6 + 7\,\sigma_2\binom{6}{4}\sigma_1^4 + \binom{7}{2}\sigma_2^2\binom{5}{2}\sigma_1^2 + \binom{7}{3}\sigma_2^3 = \\ &= 64\,\sigma_1^6 + 35\,\sigma_1^4\,\sigma_2 - 138\,\sigma_1^3\,\overline{\sigma}_3 - 139\,\sigma_1^2\,\overline{\sigma}_4 - 2\,\sigma_1\,\overline{\sigma}_5 = \\ &= 217\,[1, 6]^* + 406\,[2, 5]^* + 252\,[3, 4]^*, \end{aligned}$$

$$\begin{aligned} c_7 &= (\sigma_1^2 - 4\,\sigma_2)\,c_5 + \sigma_1^7 + 7\,\sigma_2\binom{6}{5}\sigma_1^5 + \binom{7}{2}\binom{5}{3}\sigma_1^3 + \binom{7}{3}\sigma_2^3\binom{4}{1}\sigma_1 = \\ &= 64\,\sigma_1^7 + 35\,\sigma_1^5\,\sigma_2 - 259\,\sigma_1^4\,\overline{\sigma}_3 + 315\,\sigma_1^3\,\overline{\sigma}_4 - 112\,\sigma_1^2\,\overline{\sigma}_5 = \\ &= -315\,[2, 6]^* - 350\,[3, 5]^*, \end{aligned}$$

$$\begin{aligned} c_8 &= (\sigma_1^2 - 4\,\sigma_2)\,c_6 + 7\,\sigma_2\,\sigma_1^6 + \binom{7}{2}\sigma_2^2\binom{5}{4}\sigma_1^4 + \binom{7}{3}\sigma_2^3\binom{4}{2}\sigma_1^2 + \binom{7}{4}\sigma_2^4 = \\ &= 640\,\sigma_1^8 - 4\,\sigma_1^6\,\sigma_2 - 348\,\sigma_1^5\,\overline{\sigma}_3 + 656\,\sigma_1^4\,\overline{\sigma}_4 - 655\,\sigma_1^3\,\overline{\sigma}_5 = \\ &= 245\,[3, 6]^* + 140\,[4, 5]^*, \end{aligned}$$

$$c_9 = (\sigma_1^2 - 4\,\sigma_2)\, c_7 + \binom{7}{2} \sigma_2^2\, \sigma_1^5 + \binom{7}{3} \sigma_2^3 \binom{4}{3} \sigma_1^3 + \binom{7}{4} \sigma_2^4 \binom{3}{1} \sigma_1 =$$

$$= -64\,\sigma_1^9 - 221\,\sigma_1^7\,\sigma_2 - 259\,\sigma_1^6\,\overline{\sigma}_3 + 315\,\sigma_1^5\,\overline{\sigma}_4 - 112\,\sigma_1^4\,\overline{\sigma}_5 - 140\,\sigma_1^5\,\sigma_2^2 +$$

$$+ 1036\,\sigma_1^4\,\sigma_2\,\overline{\sigma}_3 - 1260\,\sigma_1^3\,\sigma_2\,\overline{\sigma}_4 + 21\,\sigma_2^2\,\sigma_1^5 + 140\,\sigma_1^3\,\sigma_2^3 + 105\,\sigma_1\,\sigma_2^4 =$$

$$= -105\,[4\,,6]^*,$$

$$c_{10} = (\sigma_1^2 - 4\,\sigma_2)\, c_8 + \binom{7}{3} \sigma_2^3\, \sigma_2^4 + \binom{7}{4} \sigma_2^4 \binom{3}{2} \sigma_1^2 + \binom{7}{5} \sigma_2^5 =$$

$$= 64\,\sigma_1^{10} - 260\,\sigma_1^8\,\sigma_2 - 348\,\sigma_1^7\,\overline{\sigma}_3 + 656\,\sigma_1^6\,\overline{\sigma}_4 - 655\,\sigma_1^5\,\overline{\sigma}_5 + 16\,\sigma_1^6\,\sigma_2^2 +$$

$$+ 1392\,\sigma_1^5\,\sigma_2\,\overline{\sigma}_3 - 2424\,\sigma_1^4\,\sigma_2\,\overline{\sigma}_4 + 35\,\sigma_1^4\,\sigma_2^3 + 105\,\sigma_1^2\,\sigma_2^4 + 21\,\sigma_2^5 = 21\,[5\,,6]^*.$$

COROLLARY 3.4. *The canonical systems on the Grassmannian,* W = W (4 , 6), *are given by* (*where we write* [*abcde*] *for* [(*a* , *b* , *c* , *d* , *e*) ; W]:

$$X_0\,(W) = 21\,[23456]\,,$$
$$X_1\,(W) = -105\,[13456]\,,$$
$$X_2\,(W) = 245\,[12456] + 140\,[03456]\,,$$
$$X_3\,(W) = -315\,[12356] - 350\,[02456]\,,$$
$$X_4\,(W) = -217\,[12346] + 406\,[02356] + 252\,[01456]\,,$$
$$X_5\,(W) = -\;63\,[12345] - 238\,[02346] - 336\,[01356]\,,$$
$$X_6\,(W) = 57\,[02345] + 191\,[01346] + 143\,[01256]\,,$$
$$X_7\,(W) = -\;42\,[01345] - \;98\,[01246]\,,$$
$$X_8\,(W) = 22\,[01245] + \;25\,[01236]\,,$$
$$X_9\,(W) = -\;7\,[01235]\,.$$

Proof. The result follows from the Proposition since, by duality, W (4 , 6) ≅ W (1 , 6).

REFERENCES

[1] EHRESMANN C. (1934) – *Sur la topologie de certain espaces homogènes*, « Ann. of Math. » (2), *35*, 396–443.

[2] INGLETON A. W. (1969) – *Tangent flay bundles and generalized Jacobian varieties*, « Rendiconti dell'Accademia dei Lincei », 323–329, 505–510.

[3] HIRZEBRUCH F. (1966) – *Topological Methods in Algebraic Geometry*, 3rd ed., Springer Berlin.

[4] TODD J. A. (1957) – *Canonical systems on algebraic varieties*, « Bol. Soc. Mat. Mexicana », 26–44.

[5] SÉMINAIRE CHAVALLEY (2) (1958) – *Anneaux de Chow et applications*.

[6] CHERN S. S. (1953) – *On the characteristic classes of complex sphere bundles and algebraic varieties*, « An. J. Math. », *75*, 565–597.

[7] BOREL A. and HIRZEBRUCH F. (1959) – *Characteristic classes and homogeneous spaces*, I, II, « Am. J. Math. », *81*, 458–538, 315–382.

[8] MONK D. (1959) – *The geometry of flag manifolds*, « Proc. London Math. Soc. » (3), *9*, 253–286.