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Fisica matematica. — *Limiting stream-lines at a stationary boundary for isochoric motions with solenoidal acceleration.* Nota (*) di A. W. MARRIS, presentata dal Socio straniero C. TRUESDELL.

RIASSUNTO. — Si considerano i flussi isocorici di rotazionalità unitaria con aderenza alla frontiera. Si dimostra che un tale flusso è impossibile se la curvatura Gaussiana della superficie frontiera è positiva. Il teorema restringe severamente i possibili flussi di questo genere se sono piani oppure con simmetria assiale.

INTRODUCTION

I consider isochoric motion whose acceleration at any instant determines a solenoidal vector field. The motion takes place in a bounded open domain D whose partial closure is a stationary boundary surface S. The velocity magnitude is postulated to be non-vanishing in D, and to tend smoothly to zero on S, and the limiting stream-lines are tangential to the surface S. It is further postulated that the vorticity does not vanish on S. The surface S will sometimes be referred to as a surface of adherence for the motion.

Isochoric motions of solenoidal acceleration were studied by Hamel (1936 [1]). In particular he determined the plane motions of this class for a viscous fluid.

Such motions are of considerable interest because of the following. *An isochoric motion is of unit rotationality at each point if and only if its acceleration is solenoidal.*

In 1953 Truesdell (1953 [2]), after pointing out the need for some absolute measure of the rotationality of a motion, introduced the kinematical vorticity number

$$\mathfrak{W}_k = \frac{\omega}{\sqrt{2\,[d_1^2 + d_2^2 + d_3^2]}}\,, \tag{I.1}$$

where ω is the magnitude of the vorticity $\boldsymbol{\omega}$ where

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v}\,, \tag{I.2}$$

and $\boldsymbol{v}$ is the velocity. Also d_1, d_2, d_3 are the proper numbers of the deformation rate $\boldsymbol{d}$, where

$$\boldsymbol{d} = \frac{\operatorname{grad} \boldsymbol{v} + (\operatorname{grad} \boldsymbol{v})^{\mathrm{T}}}{2}\,. \tag{I.3}$$

(*) Pervenuta all'Accademia il 1° luglio 1974.

It is seen that $\mathfrak{W}_k$ is a measure of the rate of rotation relative to the rate of deformation for a particular motion [1].

Since

$$d_1^2 + d_2^2 + d_3^2 = \mathrm{I}_{\boldsymbol{d}}^2 - 2\,\mathrm{II}_{\boldsymbol{d}} \tag{I.4}$$

where $\mathrm{I}_{\boldsymbol{d}}$ and $\mathrm{II}_{\boldsymbol{d}}$ are the first and second invariants of $\boldsymbol{d}$ and since $\mathrm{I}_{\boldsymbol{d}}$ vanishes for an isochoric motion, one has

$$\mathfrak{W}_k = \frac{\omega}{\sqrt{-4\,\mathrm{II}_{\boldsymbol{d}}}}\,. \tag{I.5}$$

Also, for an isochoric motion, one has [2]

$$\operatorname{div} \boldsymbol{a} = -2\,\mathrm{II}_{\boldsymbol{d}} - \frac{\omega^2}{2} \tag{I.6}$$

where $\boldsymbol{a}$ is the acceleration. Accordingly by (I.5) and (I.6)

$$\mathfrak{W}_k = \left[1 + \frac{2 \operatorname{div} \boldsymbol{a}}{\omega^2}\right], \tag{I.7}$$

a formula which, for isochoric motions, asserts the equivalence of motions of solenoidal acceleration and motions of unit rotationality.

The formula

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} + \operatorname{grad} \frac{v^2}{2} \tag{I.8}$$

indicates that div $\boldsymbol{a}$ will be zero for any isochoric " slow motion " in which the terms $\boldsymbol{\omega} \times \boldsymbol{v}$ and grad $v^2/2$ are taken to be zero. Such motions will be of unit rotationality, and will fall into the class to be considered here. So also will isochoric screw-motions (i.e. motions for which $\boldsymbol{\omega} \times \boldsymbol{v} = 0$) satisfying the condition

$$\operatorname{grad} \frac{v^2}{2} = 0\,. \tag{I.9}$$

In the present paper I shall prove following theorem.

THEOREM I.1. *Let a smooth isochoric motion* $\boldsymbol{v}$ *endowed with solenoidal acceleration, or equivalently, of unit rotationality, be non-vanishing in a bounded open domain* D *whose partial closure is the stationary boundary surface* S, *and let the velocity magnitude tend to zero on the boundary* S, *while the vorticity is non-vanishing on* S. *The limiting stream-lines must be asymptotic lines on the boundary surface* S.

(1) The factor 2 in the denominator (I.1) is determined by considering a rectilinear shear flow, and requiring that the vorticity number reduce in this case to the ratio of the angular speed of rotation to the mean rate of shearing.

(2) This result was given by Hamel (1936 [1]). Hamel presented the analysis in terms of the invariants of grad $\boldsymbol{v}$.

This theorem limits the possible surfaces of adherence, for this class of motions, as is indicated by the following corollaries.

COROLLARY 1. *Real motions of the class postulated in the statement of Theorem I.1. are impossible if the Gaussian curvature of the boundary surface* S *is greater than zero in the region of adherence.*

This corollary follows immediately from the fact that the asymptotic lines are imaginary on a surface of positive Gaussian curvature.

COROLLARY 2. *If the postulated motion is a plane motion then a surface* S *of adherence must be a plane perpendicular to the plane of the motion.*

For a plane motion the surface of adherence must be either (*a*) a cylinder whose generators are perpendicular to the plane of the motion, or (*b*) a plane perpendicular to the plane of the motion.

Consider Case (*a*). The limiting stream-lines, being the asymptotic lines on the cylinder, are the generators of the cylinder. They are therefore perpendicular to the plane of the motion. This contradicts the postulate that the motion is plane. Case (*b*) is the only possibility. From a two-dimensional view-point the curves of adherence must be straight lines in the plane of the motion (3).

COROLLARY 3. *If the surface of adherence* S *is a surface of revolution, and if the limiting stream-lines are the meridians on this surface, then* S *must be a circular cylinder or a right circular cone* (4).

The meridians on a surface of revolution are geodesics on the surface. The principal normal to a geodesic coincides with the normal to the surface, while the principal normal to an asymptotic line lies in the tangent plane to the surface. Accordingly the meridians on the surface of revolution S can be asymptotic lines if and only if they are straight lines. This proves the corollary.

PROOF OF THEOREM I.1

In the spatial representation of a motion the acceleration is given by

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} \boldsymbol{v} = \frac{\partial v}{\partial t} + v \frac{\delta v}{\delta s} \boldsymbol{s} + \varkappa v^2 \boldsymbol{n}, \tag{1}$$

where $\boldsymbol{v}(x^\alpha, t) = v(x^\alpha, t)\, \boldsymbol{s}(x^\alpha, t)$ is the velocity at time t at the place whose co-ordinates are x^α, $\alpha = 1, 2, 3$. The unit vector $\boldsymbol{s}$ is instantaneously tangent to the stream-line and $\varkappa$ is the instantaneous stream-line curvature. For the present I shall assume that $\varkappa$ is non-vanishing in a neighbourhood, although

(3) This result is compatible with Hamel's general solution for plane motion of a viscous incompressible fluid, cited above. See (1936 [1], p. 362, Satz 11).

(4) Motions of an incompressible viscous fluid in which the stream-lines are concurrent straight lines, and which satisfy the adherence condition on a conical surface are not possible [Berker 1963 [4], Section 21, p. 52].

it will appear below that no special difficulties arise in the case when the stream-lines are rectilinear. The unit vector $\boldsymbol{n}$ denotes the instantaneous principal normal to the stream-line. It is defined where $\varkappa$ is non-vanishing. The variables $\varkappa$, $\boldsymbol{s}$ and $\boldsymbol{n}$ are each functions of x^{α} and t.

I define the instantaneous bi-normal to the stream-line according to

(2) $$\boldsymbol{b} = \boldsymbol{s} \times \boldsymbol{n},$$

and I use the notation $\frac{\delta}{\delta s}$, $\frac{\delta}{\delta n}$, $\frac{\delta}{\delta b}$ to denote components $\boldsymbol{s}\cdot$grad, $\boldsymbol{n}\cdot$grad, $\boldsymbol{b}\cdot$grad.

Consider now an isochoric motion for which $\boldsymbol{a}$ is solenoidal, so that

(3) $$\operatorname{div} \boldsymbol{v} = 0$$

and

(4) $$\operatorname{div} \boldsymbol{a} = 0.$$

From (1), (3), and (4) one has

(5) $$\begin{aligned} \operatorname{div} \boldsymbol{a} &= \frac{\partial}{\partial t} \operatorname{div} \boldsymbol{v} + \operatorname{div}\left[v \frac{\delta v}{\delta s} \boldsymbol{s} + \varkappa v^2 \boldsymbol{n}\right] \\ &= \operatorname{div}\left[v \frac{\delta v}{\delta s} \boldsymbol{s} + \varkappa v^2 \boldsymbol{n}\right] \\ &= 0. \end{aligned}$$

Since $\boldsymbol{v} = v\boldsymbol{s}$, the condition (3) can be written

(6) $$\frac{\delta v}{\delta s} + v \operatorname{div} \boldsymbol{s} = 0,$$

and one rewrites (5) in the form

(7) $$\operatorname{div}\{v^2 [(-\operatorname{div} \boldsymbol{s})\, \boldsymbol{s} + \varkappa \boldsymbol{n}]\} = 0.$$

Expanding (7), one obtains

(8) $$v^2 \left[\frac{\delta \varkappa}{\delta n} - \frac{\delta}{\delta s} \operatorname{div} \boldsymbol{s} + (\operatorname{div} \boldsymbol{s})^2 + \varkappa \operatorname{div} \boldsymbol{n}\right] + 2\, \varkappa v \frac{\delta v}{\delta n} = 0.$$

I quote from an earlier work (1970 [1]) the relations

(9) $$\frac{\delta \theta}{\delta s} + \theta^2 - \varkappa(\varkappa + \operatorname{div} \boldsymbol{n}) - \tau^2 - \Omega_n (\Omega_s - \Omega_n) = 0,$$

(10) $$\begin{aligned} \frac{\delta \varkappa}{\delta n} - \frac{\delta}{\delta s} \operatorname{div} \boldsymbol{s} + \frac{\delta \theta}{\delta s} - \varkappa^2 - (\operatorname{div} \boldsymbol{s} - \theta)^2 + (2\,\Omega - 3\,\tau)\,\tau \\ + \Omega_n (\Omega_s - \Omega_n - 4\,\tau) = 0. \end{aligned}$$

where $\Omega_s = \boldsymbol{s} \cdot \operatorname{curl} \boldsymbol{s}$ and $\Omega_n = \boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n}$ are the abnormalities of the vector-lines of $\boldsymbol{s}$ and $\boldsymbol{n}$ at the instant of consideration, τ is the torsion of the vector-line of $\boldsymbol{s}$, and θ is defined by

(11) $$\theta = \boldsymbol{b} \cdot \operatorname{grad} \boldsymbol{s} \cdot \boldsymbol{b}.$$

By virtue of (9) and (10) one may write (8) as

$$v^2 \{-(\operatorname{div} s)^2 + \theta(\operatorname{div} s - \theta) + (\Omega_n + \tau)[\Omega_s - (\Omega_n + \tau)]\} - \varkappa v \frac{\delta v}{\delta n} = 0. \tag{12}$$

Alternatively one may determine the deformation rate $\boldsymbol{d}$ from the formulae of 1970 [4], and compute $\mathrm{II}_{\boldsymbol{d}}$. Recalling the formula for the vorticity

$$\boldsymbol{\omega} = \Omega v \boldsymbol{s} + \frac{\delta v}{\delta b} \boldsymbol{n} + \left(\varkappa v - \frac{\delta v}{\delta n}\right) \boldsymbol{b}, \tag{13}$$

one readily verifies that the condition (13) is indeed a representation of the relation

$$4\,\mathrm{II}_{\boldsymbol{d}} + \omega^2 = 0, \tag{14}$$

the condition required by (1.6) and (4).

The motion is postulated to take place in a bounded open domain D whose partial closure is a surface S. The motion is smooth so that all the terms in (12) are bounded and continuous in D. The velocity magnitude v is postulated to be non-vanishing in D, but to tend smoothly to zero on the boundary surface S. The vorticity does not vanish on S. The limiting stream-lines are tangential to the surface S.

In the domain D, and indeed, arbitrarily close to the boundary S one may cancel v from (12) and write

$$v\{-(\operatorname{div} \boldsymbol{s})^2 + \theta(\operatorname{div} \boldsymbol{s} - \theta) + (\Omega_n + \tau)[\Omega_s - (\Omega_n + \tau)]\} - \varkappa \frac{\delta v}{\delta n} = 0. \tag{15}$$

By continuity (15) will hold on the boundary S itself. Accordingly, on the boundary surface S, one must have

$$\varkappa \frac{\delta v}{\delta n} = 0. \tag{16}$$

It appears that upon the boundary surface S, either 1) $\varkappa = 0$, or 2) $\frac{\delta v}{\delta n} = 0$.

When $\varkappa$ vanishes the limiting stream-lines are rectilinear, so that the boundary surface S must be a ruled surface with the limiting stream-lines as its rectilinear generators. The limiting stream-lines are at once geodesics and asymptotic lines on the boundary surface.

To consider the case $\frac{\delta v}{\delta n} = 0$, I note from (13) that, on the boundary surface S, on which v vanishes, the vorticity has the value

$$\boldsymbol{\omega} = \frac{\delta v}{\delta b} \boldsymbol{n}, \quad \text{where} \quad \frac{\delta v}{\delta b} \neq 0. \tag{17}$$

Accordingly, on the boundary surface the vorticity vector is parallel to the principal normal to the limiting stream-line. Application of Kelvin's transformation (Stokes' theorem) to a circuit situated on the boundary surface shows that the vorticity vector $\boldsymbol{\omega}$ must lie in the tangent plane to the boundary surface. It follows from (17) that the principal normal $\boldsymbol{n}$ to the limiting stream-line must lie in the tangent plane to the boundary surface S. The bi-normal to the limiting stream-line is thus normal to the boundary surface S. The limiting stream-lines must be asymptotic lines on the surface S.

I postulated originally that the motion was non-rectilinear in the neighbourhood so that the instantaneous principal normal to the stream-line could be defined immediately. If the motion is rectilinear in D, then by (3) and the representation theorem for solenoidal vector fields one may write

$$\boldsymbol{v} = v\boldsymbol{s} = \operatorname{grad} \alpha \times \operatorname{grad} \beta. \tag{18}$$

It follows that either $\boldsymbol{s} \cdot \operatorname{grad} \alpha = 0$ or $\boldsymbol{s} \cdot \operatorname{grad} \beta = 0$. One may choose $\boldsymbol{n}$ to be that normal to the stream-line which is parallel to either one of grad α or grad β. The analysis then carries through as before, with Case 1 representing the only possibility. This completes the proof of Theorem (I.1).

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