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An axiomatic topological characterization of an uncountable product of real lines

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Topologia. — *An axiomatic topological characterization of an uncountable product of real lines.* Nota di JOHAN SWART (*), presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà condizioni necessarie e sufficienti affinché uno spazio topologico sia omeomorfo al prodotto di un'infinito non-numerabile di rette.

In [5] the Author gave an axiomatic topological characterization of R^n (n-dimensional Euclidean space) and of R^∞ (countable infinite product of real lines) based on ideas and methods used by J. de Groot in his characterization of the n-cell, I^n, and the Hilbert cube, I^∞ (see [3]). In view of Anderson's well known result, namely that Hilbert space, l_2, is homeomorphic to the countable infinite product of lines (see [1] and [2]) a topological characterization of l_2 was obtained. The purpose of this paper is to complete the topological characterization of arbitrary products of lines by giving a topological characterization of an uncountable product of lines.

A topological space X is said to be 2-*compact* if there exists an open subbase for X such that every open cover of X by subbase elements has a subcover by two elements. A family $\mathcal{S}$ of subsets of X is called *comparable* if for all $S_0, S_1, S_2 \in \mathcal{S}$, whenever $S_0 \cup S_1 = X$ and $S_0 \cup S_2 = X$, then $S_1 \subseteqq S_2$ (i.e. $S_1 \subset S_2$ or $S_2 \subset S_1$). If X has a comparable open subbase $\mathcal{S}$ relative to which X is 2-compact then X is said to be 2-*ccompact*. The above concepts are all due to J. de Groot (see [3] and [4]).

Consider now an arbitary product of lines πR_α. The open subbase $\mathcal{G}$ of πR_α consisting of all sets of the form $p_\alpha^{-1}(-\infty, a)$ and $p_\alpha^{-1}(a, \infty)$, $a \in R$ and all α, is comparable and furthermore has the property that every open cover of the space which contains at least one member of the form $p_\alpha^{-1}(-\infty, a)$ and one of the form $p_\alpha^{-1}(a, \infty)$ for each α has a subcover consisting of two of these sets. It is therefore possible to formulate a "multi-2-compactness" condition in terms of open covers containing non-empty members of each of the linearly ordered (by inclusion) subfamilies of $\mathcal{G}$. This observation is basic to the topological characterization of products of lines.

The theorem below is stated in terms of closed subbases. By a *linked* family of subsets of X we shall mean a family with the property that every pair of members has non-empty intersection. $\mathcal{E}$ will denote the class of all linearly ordered (by inclusion) subfamilies E of the subbasis $\mathcal{S}$.

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THEOREM. *A topological space* (X, τ) *is homeomorphic to* $\pi\{R_\alpha \mid \alpha \in A$, *card* $A > \aleph_0\}$ *if and only if*

(1) X *is* T_1;

(2) X *is connected*;

(3) *There exists a closed subbasis* $\mathcal{S}$ *for* X *which does not contain* X *nor* φ *and which satisfies* (4), (5), (6) and (7);

(4) $\mathcal{S}$ *is comparable*, i.e. $\forall S_0, S_1, S_2 \in \mathcal{S}$,

$$\left.\begin{array}{l} S_0 \cap S_1 = \varphi \\ S_0 \cap S_2 = \varphi \end{array}\right\} \Rightarrow S_1 \subsetneqq S_2 ;$$

(5) *Each* $E \in \mathcal{E}$ *is countable and satisfies*

$$\cap E = \varphi \quad \text{and} \quad \cup E = X ;$$

(6) *Every linked* $\mathcal{F} \subset \mathcal{S}$ *which satisfies*

$$E \cap \mathcal{F} = \varphi \quad \text{and} \quad E \not\subset \mathcal{F} \quad \forall E \in \mathcal{E}$$

has the property that $\cap \mathcal{F} \neq \varphi$;

(7) card $\mathcal{E}$ = card A.

The proof is similar to that in [5] and, in brief outline, consists of showing that the linearly ordered subfamilies of X are equivalence classes induced by the comparability relation « $\subsetneqq$ », that these occur in conjugate pairs (E_α, E'_α), $\alpha \in A$ and then constructing for each $\alpha \in A$ a continuous surjection $f_\alpha : X \rightarrow R_\alpha$ such that $E_\alpha = \{f_\alpha^{-1}(-\infty, r] \mid r$ is a rational number$\}$. X may then be homeomorphically embedded in $\pi\{R_\alpha \mid \alpha \in A\}$ by means of the diagonal map $\Delta : X \rightarrow \pi R_\alpha$

$$x \mapsto (f_\alpha(x))_{\alpha \in A}$$

REFERENCES

[1] R. D. ANDERSON (1966) – *Hilbert space is homeomorphic to the countable infinite product of lines*, « Bull. Amer. Math. Soc. », 72, 515–519.

[2] R. D. ANDERSON and R. H. BING (1968) – *A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines*, « Bull. Amer. Math. Soc. », 74, 771–729.

[3] J. DE GROOT (1972) – *Topological characterization of metrizable cubes*, in: Felix Hausdorff Gedenkband (VEB Deutscher Verlag der Wissenschaften).

[4] J. DE GROOT and P. S. SCHNARE (1972) – *A topological characterization of products of compact totally ordered spaces*, « Gen. Topology Appl. », 2, 67–73.

[5] J. SWART (1972) – *An axiomatic topological characterization of Hilbert space*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. », 52, 166–174.