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Morphisms of affine Hjelmslev planes

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Geometria. — *Morphisms of affine Hjelmslev planes.* Nota di JOSEPH W. LORIMER e NORMAN D. LANE, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si stabiliscono varie proprietà dei morfismi fra piani affini di Hjelmslev e caratterizzazioni degli isomorfismi fra quelli.

1. INTRODUCTION

If $\mathbf{P}$ and $\mathbf{L}$ are sets and $I \subseteq \mathbf{P} \times \mathbf{L}$ ($\| \subseteq \mathbf{L} \times \mathbf{L}$ is an equivalence relation), then $\mathscr{S} = \langle \mathbf{P}, \mathbf{L}, I \rangle$ ($\langle \mathbf{P}, \mathbf{L}, I, \| \rangle$) is an incidence structure (with parallelism). If $\mathscr{S}_1$ and $\mathscr{S}_2$ are two incidence structures (with parallelism), then a morphism from $\mathscr{S}_1$ to $\mathscr{S}_2$ is a pair (Φ, Ψ), where Φ maps $\mathbf{P}_1$ to $\mathbf{P}_2$, Ψ maps $\mathbf{L}_1$ to $\mathbf{L}_2$, and incidence (and parallelism) is preserved.

Various authors have considered the conditions under which these morphisms are isomorphisms in special classes of incidence structures; cfr. André [1], Satz 3.1; Artmann [2], 1.1; Cronheim [5], p. 2; Dembowski [6]; and Corbas [4].

In this paper, we examine the above problem for affine Hjelmslev planes.

1.1. NOTATION. Let $\langle \mathbf{P}, \mathbf{L}, I, \| \rangle$ be an incidence structure with parallelism. The elements of $\mathbf{P}$ [$\mathbf{L}$] are *points* [*lines*] and are denoted by $P, Q, \cdots$ [$l, m, \cdots$]. We write $l \| m$ for $(l, m) \in \|$ and PIl for $(P, l) \in I$. P, QIl shall mean PIl and QIl. We put $g \wedge h = \{P \in \mathbf{P} \mid PIg, h\}$. If $A \subset \mathbf{P}$, $|A|$ is the cardinality of the set A. Define $(P, Q) \in \sim_{\mathbf{P}}$ if there exist $l, m \in \mathbf{L}$, $l \neq m$, such that P, QIl, m. We usually write $P \sim Q$ for $(P, Q) \in \sim_{\mathbf{P}}$. Define $(l, m) \in \sim_{\mathbf{L}}$ (or $l \sim m$) if for every PIl there exists QIm such that $P \sim Q$ and for every QIm there exists PIl such that $Q \sim P$. If $P \sim Q$ [$l \sim m$] we call P and Q [l and m] *neighbours*. If P and Q [l and m] are not neighbours, we write $P \nsim Q$ [$l \nsim m$].

An *affine Hjelmslev plane* $\mathscr{H} = \langle \mathbf{P}, \mathbf{L}, I, \| \rangle$, is an incidence structure with parallelism, which satisfies the following system of axioms.

(A 1) For any two points P and Q there exists $l \in \mathbf{L}$ such that P, QIl. We write $l = PQ$ if $P \nsim Q$;

(A 2) There exist $P_1, P_2, P_3 \in \mathbf{P}$ such that $P_i P_j \nsim P_i P_k$; $i \neq j \neq k \neq i$; $i, j, k = 1, 2, 3$;

(A 3) $\sim_{\mathbf{P}}$ is transitive on $\mathbf{P}$;

(A 4) If PIg, h, then $g \nsim h$ iff $|g \wedge h| = 1$;

(A 5) If $g \nsim h$; P, RIg; Q, RIh; and $P \sim Q$, then $R \sim P, Q$;

(*) Nella seduta del 29 giugno 1974.

(A 6) If $g \sim h$; $j \nsim g$; PI$g, j$; and QI$h, j$; then $P \sim Q$;

(A 7) If $g \parallel h$; PI$j, g$; and $g \nsim j$; then $j \nsim h$ and there exists Q such that QI$h, j$;

(A 8) For every $P \in \mathbf{P}$ and every $l \in \mathbf{L}$, there exists a unique line $L(P, l)$ such that PIL(P, l) and $l \parallel L(P, l)$.

Let $\mathscr{H} = \langle \mathbf{P}, \mathbf{L}, I, \parallel \rangle$ be an affine Hjelmslev plane, henceforth called an A. H. plane. If Π and Π' are pencils of parallel lines, we define $\Pi \sim \Pi'$ if each line of Π is a neighbour of some line of Π'. This is an equivalence relation. Let Π_l be the pencil of lines parallel to l. Then $\Pi_l \nsim \Pi_m$ if and only if $| l \wedge m | = 1$; cfr. [9], Satz 2.9.

With each A. H. plane $\mathscr{H}$ there is associated an ordinary affine plane $\bar{\mathscr{H}} = \langle \bar{\mathbf{P}}, \bar{\mathbf{L}}, \bar{I} \rangle$. Here, $\bar{\mathbf{P}}$ and $\bar{\mathbf{L}}$ are the quotient spaces of $\sim_{\mathbf{P}}$ and $\sim_{\mathbf{L}}$, respectively, and $\bar{P}\bar{I}\bar{l}$ if there exists SIl such that $S \sim P$. Let $\chi_{\mathbf{P}}$ and $\chi_{\mathbf{L}}$ be the quotient maps of $\sim_{\mathbf{L}}$ and $\sim_{\mathbf{P}}$, respectively: cfr. [9], Satz 2.6.

If $l \in \mathbf{L}$, there exist P, QIl such that $P \nsim Q$, and if $P \in \mathbf{P}$, there exist $l, m \in \mathbf{L}$ such that PI$l, m$ and $l \nsim m$; cfr. [9], Satz 2.3 and 2.4.

1.2. MORPHISMS (cfr. [7], 1.2, 1.3). Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be A. H. planes.

(*a*) $f = (\Phi, \Psi) : \mathscr{H}_1 \to \mathscr{H}_2$ is a *morphism* from $\mathscr{H}_1$ to $\mathscr{H}_2$ if the following conditions hold.

(i) $\Phi : \mathbf{P}_1 \to \mathbf{P}_2$ and $\Psi : \mathbf{L}_1 \to \mathbf{L}_2$ are maps;

(ii) $\Phi(P_1)\, I_2\, \Psi(l_1)$ whenever $P_1\, I_1\, l_1$;

(iii) $\Psi(l)_1 \parallel_2 \Psi(m_1)$ whenever $l_1 \parallel_1 m_1$.

In general, we shall write I, $\parallel$, and L for I_i, $\parallel_i$ and L_i, respectively, for $i = 1, 2$, unless ambiguity arises.

If f satisfies only (i) and (ii) we shall call f an incidence morphism, or an I-morphism. f is a *neighbour-preserving* I-morphism if $P \sim Q$ implies $\Phi P \sim \Phi Q$ and $l \sim m$ implies $\Psi l \sim \Psi m$; cfr. [8], p. 136.

(*b*) $f = (\Phi, \Psi)$ is an *epimorphism* if Φ and Ψ are both surjective.

(*c*) $f = (\Phi, \Psi)$ is a *monomorphism* if Φ and Ψ are both injective.

(*d*) $f = (\Phi, \Psi)$ is an I-*isomorphism* if f is an I-morphism such that Φ and Ψ are bijective and $P_1\, I_1\, l_1 \Longleftrightarrow \Phi(P_1)\, I_2\, \Psi(l_1)$. If, in addition, $l_1 \parallel_1 m_1 \Longleftrightarrow \Psi l_1 \parallel_2 \Psi m_1$ then f is an *isomorphism*. If $\mathscr{H}_1 = \mathscr{H}_2$, then f is an *automorphism*.

Remark. For ordinary affine planes, the concepts of an I-isomorphism and an isomorphism are identical; cfr. 2.3. However, P. Bacon has constructed an I-isomorphism between two A. H. planes which is not an isomorphism; cfr. [3], Corollaries 3.11 and 3.12.

2. Morphisms of ordinary affine planes

We shall first consider the special case where $\mathscr{H}_1 = \mathscr{A}_1$ and $\mathscr{H}_2 = \mathscr{A}_2$ are ordinary affine planes, and $f = (\Phi, \Psi)$ is an-morphism. In this case, the analysis of morphisms is made easier due to fact that parallelism is defined in terms of incidence. Let I.P. denote the property $\mathrm{PI}l$ if and only if $\Phi(\mathrm{P})$ $\mathrm{I}\Psi(l)$. By using the methods of V. Corbas in [4], one can easily verify the following statements.

2.1. LEMMA. (1) *If* Ψ *is injective, then* $l \parallel m$ *whenever* $\Psi(l) \parallel \Psi(m)$;

(2) *If* f *is a morphism and* Ψ *is surjective, then* Φ *is surjective*;

(3) *If* Φ *is surjective and* f *has* I.P., *then* f *is a morphism*;

(4) f *has* I.P *if and only if* f *is an* I-*monomorphism*;

(5) *If* Φ *surjective, then* Ψ *is surjective.*

2.2. The main result of Corbas in [4] is the following assertion. *If* f *is an* I-*epimorphism, then* Φ *and* Ψ *are both injective.*

2.3. From 2.1 and 2.2, we readily obtain the following characterizations of an isomorphism.

THEOREM. *Let* $f = (\Phi, \Psi) : \mathscr{A}_1 \to \mathscr{A}_2$ *be an* I-*morphism. Then the following are equivalent*:

(1) f *is an isomorphism*;

(2) f *is an* I-*isomorphism*;

(3) f *is an* I-*epimorphism*;

(4) Φ *is surjective*;

(5) Ψ *is surjective and* f *is a morphism.*

3. Morphisms of A. H. planes

3.1. The objective of our paper is the proof of the following result.

THEOREM. *Let* $f = (\Phi, \Psi) : \mathscr{H}_1 \to \mathscr{H}_2$ *be a morphism. The the following statements are equivalent*:

(1) f *is an isomorphism*;

(2) Φ *and* Ψ *are bijective*;

(3) Φ *is surjective and* Ψ *is injective*;

(4) Φ *is surjective,* $l \parallel m$ *whenever* $\Psi(l) \parallel \Psi(m)$, *and* f *is neighbour-preserving.*

For the proof of our theorem, we first establish some preliminary lemmas.

3.2. LEMMA. *Let* $f = (\Phi, \Psi) : \mathscr{H}_1 \to \mathscr{H}_2$ *be an* I-*morphism. If* Φ *is surjective then* ψ *is surjective.*

Proof. Let l_2 in $\mathscr{H}_2$. Choose P_2 and Q_2 on l_2 such that $P_2 \nsim Q_2$. Then there exist distinct points P_1 and Q_1 in $\mathscr{H}_1$ such that $\Phi(P_1) = P_2$ and $\Phi(Q_1) = Q_2$. Select any line l_1 through P_1 and Q_1. Since $P_1, Q_1 \, I l_1$, we have $P_2, Q_2 \, I \Psi(l_1)$. Hence $\Psi(l_1) = l_2$.

3.3. LEMMA. *Let* $f = (\Phi, \Psi) : \mathscr{H}_1 \to \mathscr{H}_2$ *be a morphism. The the following statements are valid*:

(1) *If* Ψ *is injective, then* $P \sim Q$ *implies* $\Phi(P) \sim \Phi(Q)$;

(2) $\Psi(L(P, l)) = L(\Phi(P), \Psi(l))$;

(3) *If* $P \nsim Q$ *and* $\Phi(P) \nsim \Phi(Q)$, *then* $\Psi(PQ) = \Phi(P)\Phi(Q)$;

(4) *If* $\Pi_l \nsim \Pi_m$ *and* $\Pi_{\Psi(l)} \nsim \Pi_{\Psi(m)}$, *then* $\Phi(l \wedge m) = \Psi(l) \wedge \Psi(m)$.

3.4. LEMMA. *Let* $f = (\Phi, \Psi) : \mathscr{H}_1 \to \mathscr{H}_2$ *be a morphism such that* Φ *is surjective and* Ψ *is injective. Then*

(1) $\{Q \mid QI\Psi(l)\} = \{\Phi(P) \mid PIl\}$;

(2) *If* $l \sim m$, *then* $\Psi(l) \sim \Psi(m)$.

Proof. (1) Since f is a morphism, $\{\Phi(P) \mid PIl\} \subseteq \{Q \mid QI\Psi(l)\}$. Now take $QI\Psi(l)$. Then there exists P such that $\Phi(P) = Q$. Now $\Psi(l) = L(\Phi(P), \Psi(l)) = \Psi(L(P, l))$, by Lemma 3.3. Since Ψ is injective, $l = L(P, l)$ and so PIl.

(2) Let $l \sim m$. Choose $RI\Psi(l)$. By (1), $R = \Phi(P)$ for some point PIl. Then there exists QIm such that $P \sim Q$. By Lemma 3.2, $\Phi(P) \sim \Phi(Q)$. Hence $\Psi(l) \sim \Psi(m)$.

Similarly, every point of $\Psi(m)$ is a neighbour of some point of $\Psi(l)$.

3.5. LEMMA. *Let* $f = (\Phi, \Psi) : \mathscr{H}_1 \to \mathscr{H}_2$ *be a morphism. Then the following statements are equivalent*:

(1) PIl *if and only if* $\Phi(P) I \Psi(l)$;

(2) f *is a monomorphism*;

(3) Ψ *is injective.*

Proof. To show that (1) implies (2), we shall first prove that Φ is injective. Suppose that $P \neq Q$. Then we may choose l such that PIl but $Q \not\!I l$. By (1), $\Phi(P) I \Psi(l)$ but $\Phi(Q) \not\!I \Psi(l)$. Hence $\Phi(P) \neq \Phi(Q)$. Similarly, we can verify that Ψ is injective.

It is obvious that (2) implies (3). Finally, we shall show that (3) implies (1). Let Ψ be injective. Suppose that $\Phi(P) I \Psi(l)$. Then $\Phi(l) = L(\Phi(P), \Psi(l)) = \Psi(L(P, l))$, by Lemma 3.3. Since Ψ is injective, $l = L(P, l)$ and so PIl.

3.6. LEMMA. *Let* $f = (\Phi, \Psi) : \mathscr{H}_1 \to \mathscr{H}_2$ *be a morphism such that* Ψ *is injective. Then* $l \parallel m$ *whenever* $\Psi(l) \parallel \Psi(m)$.

Proof. Assume that $\Psi(l) \parallel \Psi(m)$. If $l \neq m$, then there exists PIl such that $P \not\!I m$. Put $j = L(P, m)$; thus $j \parallel m$. Then $\Psi(j) \parallel \Psi(m)$ and so $\Psi(j) \parallel \Psi(l)$. But PIj, l implies $\Phi(P) I \Psi(j), \Psi(l)$, and so $\Psi(j) = \Psi(l)$. Since Ψ is injective, $j = l$ and so $l \parallel m$.

3.7. *Remark.* In 1.2, we require an isomorphism to have the properties

$$\mathrm{P I} l \Longleftrightarrow \Phi(\mathrm{P})\, \mathrm{I} \Psi(l) \quad \text{and} \quad l \parallel m \Longleftrightarrow \Psi(l) \parallel \Psi(m)\,.$$

Lemmas 3.5 and 3.6 show that this definition is redundant with respect to both incidence and parallelism.

3.8. Let Aut $\mathscr{H}$ and Aut $\overline{\mathscr{H}}$ denote the groups of automorphisms of $\mathscr{H}$ and $\overline{\mathscr{H}}$; cfr. 1.1.

We call an automorphism $f = (\Phi, \Psi)$ of Aut $\mathscr{H}$ a *neighbouring automorphism* if $\Phi(\mathrm{P}) \sim \mathrm{P}$ and $\Psi(l) \sim l$ for each P and l. The set of neighbouring automorphisms shall be denoted by N Aut $\mathscr{H}$.

In view of Lemma 3.4, if $f \in$ Aut $\mathscr{H}$, we may put $f = (f, f)$. We shall establish a relationship between Aut $\mathscr{H}$ and Aut $\overline{\mathscr{H}}$.

3.9. THEOREM. *The map* $h : \mathrm{Aut}\ \mathscr{H} \to \mathrm{Aut}\ \overline{\mathscr{H}} (f \to \bar{f})$, *where* $\bar{f}(\overline{\mathrm{P}}) = \overline{f(\mathrm{P})}$ *and* $\bar{f}(\bar{l}) = \overline{f(l)}$ *for any* $\mathrm{P} \in \mathbf{P}$ *and any* $l \in \mathbf{L}$, *is a group homomorphism and* $\chi \circ f = \bar{f} \circ \chi$. *Moreover,* N Aut $\mathscr{H}$ *is the kernel of h and*

$$\mathrm{Aut}\ \mathscr{H} / \mathrm{N\ Aut}\ \mathscr{H} \cong h\,[\mathrm{Aut}\ \mathscr{H}]\,.$$

Proof. We first show that $\bar{f}$ is well-defined on $\mathbf{P}$. Let $\overline{\mathrm{P}} = \overline{\mathrm{Q}}$. Then $\mathrm{P} \sim \mathrm{Q}$, and so $f(\mathrm{P}) \sim f(\mathrm{Q})$, by Lemma 3.3. Hence $\bar{f}(\overline{\mathrm{Q}}) = \overline{f(\mathrm{Q})} = \overline{f(\mathrm{P})} = \bar{f}(\overline{\mathrm{P}})$. Similarly, Lemma 3.4 shows that $\bar{f}$ is well-defined on $\mathbf{L}$. Now we show that $\bar{f} \in$ Aut. $\overline{\mathscr{H}}$. Let $\overline{\mathrm{P}}\, \bar{\mathrm{I}}\, \bar{l}$. Then there exists $\mathrm{SI}l$ such that $\mathrm{S} \sim \mathrm{P}$. Hence $f(\mathrm{S})\, \mathrm{I}\, f(l)$ and $f(\mathrm{S}) \sim f(\mathrm{P})$ and so $\overline{f(\mathrm{P})}\, \bar{\mathrm{I}}\, \overline{f(l)}$. By definition, $\bar{f}$ is surjective. Then by 2.3, $\bar{f} \in$ Aut $\overline{\mathscr{H}}$. Next.

$$(\chi \circ f)(\mathrm{P}) = \chi(f(\mathrm{P})) = \overline{f(\mathrm{P})} = \bar{f}(\overline{\mathrm{P}}) = (\bar{f} \circ \chi)(\mathrm{P})$$

and

$$(\overline{f \circ g})(\overline{\mathrm{P}}) = \overline{(f \circ g)(\mathrm{P})} = \bar{f}(\overline{g(\mathrm{P})}) = (\bar{f} \circ \bar{g})(\overline{\mathrm{P}}).$$

Hence h is a homomorphism. Finally, $f \in$ Ker h if and only if $\overline{f(\mathrm{P})} = \overline{\mathrm{P}}$ and $\overline{f(l)} = \bar{l}$ if and only if $f(\mathrm{P}) \sim \mathrm{P}$ and $f(l) \sim l$.

3.10. LEMMA. *If* $f : \mathscr{H}_1 \to \mathscr{H}_2$ *is a neighbour-preserving* I-*epimorphism, then* $\bar{f} : \overline{\mathscr{H}_1} \to \overline{\mathscr{H}_2}$, *well-defined as in 3.9 by* $\overline{\Phi}(\overline{\mathrm{P}_1}) = \overline{\Phi(\mathrm{P}_1)}$ *and* $\overline{\Psi}(\bar{l}_1) = \overline{\Psi(l_1)}$, *is also an* I-*epimorphism. Hence by 2.3,* $\bar{f}$ *is an isomorphism.*

Proof. Let $\overline{\mathrm{P}}_2 \in \overline{\mathscr{H}_2}$; thus $\mathrm{P}_2 \in \mathscr{H}_2$. Since Φ is surjective, there exists $\mathrm{P}_1 \in \mathscr{H}_1$ such that $\Phi(\mathrm{P}_1) = \mathrm{P}_2$. Then $\overline{\Phi}(\overline{\mathrm{P}}_1) = \overline{\Phi(\mathrm{P}_1)} = \overline{\mathrm{P}}_2$. Similarly, $\overline{\Psi}$ is surjective. Finally, let $\overline{\mathrm{P}}_1\, \bar{\mathrm{I}}\, \bar{l}_1$; thus there exists $\mathrm{S}_1 \in \mathscr{H}_1$ such that $\mathrm{S}_1\, \mathrm{I}\, l_1$ and $\mathrm{S}_1 \sim \mathrm{P}_1$. Then $\Phi(\mathrm{S}_1)\, \mathrm{I}\, \Psi(l_1)$ and so $\overline{\Phi(\mathrm{S}_1)}\, \overline{\mathrm{I}\Psi(l_1)}$; i.e., $\overline{\Phi}(\overline{\mathrm{S}}_1)\, \bar{\mathrm{I}}\, \overline{\Psi}(\bar{l}_1)$. Since $\overline{\mathrm{P}}_1 = \overline{\mathrm{S}}_1$, we have $\overline{\Phi}(\overline{\mathrm{P}}_1)\, \bar{\mathrm{I}}\, \overline{\Psi}(\bar{l}_1)$.

3.11. LEMMA. *If* $f : \mathscr{H}_1 \to \mathscr{H}_2$ *is a neighbour-preserving* I-*epimorphism, then*

(1) $\mathrm{P} \sim \mathrm{Q} \Longleftrightarrow \Phi(\mathrm{P}) \sim \Phi(\mathrm{Q})$;

(2) $l \sim m \Longleftrightarrow \Psi(l) \sim \Psi(m)$.

Proof. (1) By 3.10, $\bar{f}$ is an isomorphism, and so

$$P \nsim Q \Longleftrightarrow \bar{P} \neq \bar{Q} \Longleftrightarrow \bar{\Phi}(\bar{P}) \neq \bar{\Phi}(\bar{Q}) \Longleftrightarrow \overline{\Phi(P)} \neq \overline{\Phi(Q)} \Longleftrightarrow \Phi(P) \nsim \Phi(Q).$$

We can verify (2) in a similar fashion.

3.12. *Proof of Theorem 3.1.* Clearly, (1) implies (3). To show that (3) implies (2), we need only to verify that Ψ is surjective, since Φ is injective by Lemma 3.5. Choose $l_2 \in \mathbf{L}_2$. By 1.1, we can choose $P_2, Q_2 \, I l_2$ such that $P_2 \nsim Q_2$. Then there exist P_1 and Q_1 such that $\Phi(P_1) = P_2$ and $\Phi(Q_1) = Q_2$. By Lemma 3.3, $P_1 \nsim Q_1$. By Lemma 3.3, again $\Psi(P_1, Q_1) = \Phi(P_1)\,\Phi(Q_1) = P_2 Q_2 = l_2$. Next, (2) implies (4), by 3.3, 3.4 and 3.6. Finally we show that (4) implies (1). By 3.2, f is an epimorphism. Let $P, Q \in \mathscr{H}_1$, $P \neq Q$. If $P \nsim Q$, then $\Phi P \nsim \Phi Q$, by 3.11, and so $\Phi P \neq \Phi Q$. Suppose now that $P \neq Q$ but $P \sim Q$. Choose a line l through P such that l is not a neighbour of any line through P and Q; cfr. 1.1. Thus $Q \not{I} l$. Select a point $R I l$ such that $R \nsim P$. Then $R \nsim Q$, by (A3), and by 3.11, $\Phi R \nsim \Phi P, \Phi Q$. As $RP \neq RQ$, we have $RP \nparallel RQ$. Hence $\Psi(RP) \nparallel \Psi(RQ)$, and by 3.3, $\Phi R\,\Phi P \nparallel \Phi R\,\Phi Q$. Hence $\Phi P \neq \Phi Q$. Thus Φ is injective.

Next we wish to show that Ψ is injective. Let $l, m \in \mathscr{H}_1$, $l \neq m$. If $l \nparallel m$, then $\Psi l \nparallel \Psi m$, and $\Psi l \neq \Psi m$. Next, suppose $l \neq m$ and $l \parallel m$. Choose $P I l$ and $j \nsim l$ such that $P I j$. By (A7), $j \nsim m$, and there is a point $Q I m, j$; and $Q \neq P$. Since Φ is injective $\Phi P \neq \Phi Q$. By 3.11, $\Psi j \nsim \Psi l, \Psi m$. Since $\Phi P I \Psi l, \Psi j$; and $\Phi Q I \Psi m, \Psi j$, we obtain $\Psi l \neq \Psi m$, otherwise Ψl would be a neighbour of Ψj.

Assertion (1) of Theorem 3.1 now follows from 3.5.

Remark. The Authors have show that an automorphism of a Desarguesian A.H. plane $\mathscr{H}$ with a coordinate ring H can be represented by a non-singular semi-linear transformation of the left module structure on $H \times H$. This result can also be derived by embedding $\mathscr{H}$ in the projective Hjelmslev space over the free module $H \times H \times H$: cfr. ([8], 2 and 8).

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