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On periodic solutions of a functional equation

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Equazioni funzionali nel campo complesso. — *On periodic solutions of a functional equation.* Nota di FRED GROSS e CHUNG-CHUN YANG, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota è data la forma delle soluzioni intere periodiche $H_i(z)$, $(i = 1, 2, \cdots, k)$ dell'equazione funzionale $\sum_{i=1}^{k} H_i(z)\, e^{\alpha_i z} = 0$ dove le α_i sono costanti e i periodi delle $H_i(z)$ soddisfano la condizione che τ_i/τ_j non è reale per $i \neq j$.

In [1] a necessary and sufficient condition is provided in order that the following functional equation:

$$H_1(z)\, e^{\alpha(z)} = H_2(z) \tag{1}$$

have periodic meromorphic function solutions $H_1(z)$ and $H_2(z)$ with periods τ_1 and τ_2 respectively. Here $\alpha(z)$ is a given entire function and τ_1, τ_2 are required to be linearly independent over the reals, i.e., $\tau_1/\tau_2 \neq$ real.

It is then natural for one to study the following more general type of equation:

$$\sum_{i=1}^{n} H_i(z)\, e^{\alpha_i(z)} = 0 \tag{2}$$

where $H_1, H_2, \cdots, H_n$ are required to be periodic entire functions with periods $\tau_1, \tau_2, \cdots, \tau_n$, respectively, obeying the conditions: $\tau_i/\tau_j \neq$ real whenever $i \neq j$, and $\alpha_1(z), \alpha_2(z), \cdots, \alpha_n(z)$ are given entire functions.

In this Note, we shall present the following result which is related to the general form of the periodic entire solutions of equation (2) when all the functions $\alpha_i(z)$, $i = 1, 2, \cdots, n$ are linear functions.

THEOREM. *Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n given complex numbers $(n \geq 2)$ and $H_1(z), H_2(z), \cdots, H_n(z)$ be n periodic entire functions with periods $\tau_1, \tau_2, \cdots, \tau_n$, respectively, obeying the restrictions: $\tau_i/\tau_j \neq$ real whenever $i \neq j$. Suppose that $H_1(z), H_2(z), \cdots, H_n(z)$ satisfy the following functional equation:*

$$\sum_{i=1}^{n} H_i(z)\, e^{\alpha_i z} = 0\,. \tag{3}$$

then

$$H_i(z) = \sum_{i=1}^{n-1} b(i, j)\, e^{l(i,j)\gamma_i z}$$

for $i = 1, 2, \cdots, n$ where $b(i, j)$ is a constant, $l(i, j)$ is an integer, and γ_i is constant.

(*) Nella seduta del 29 giugno 1974.

We shall first prove the statement for the case when $n = 2$, and then proceed by mathematical induction to prove the statement in general. The following lemma is our key to the whole proof.

LEMMA. *Let* α *be a constant and* $H_1(z)$ *and* $H_2(z)$ *be two periodic entire functions with periods* τ_1 *and* τ_2*, respectively, and* $\tau_1/\tau_2 \neq$ *real. Suppose that* $H_1(z)$ *and* $H_2(z)$ *satisfy the equation*

$$H_1(z)\, e^{\alpha z} = H_2(z).$$

Then $H_1(z) = C_1 e^{\beta z}$ and $H_2(z) = C_2 e^{\gamma z}$ where C_1, C_2, β and γ are constants.

This lemma is a special case of Theorem 2 in [1].

Proof of the Theorem. It follows immediately from the key lemma that when $n = 2$ the statement of our theorem is valid. Now we assume the statement is true for $n \leq k - 1$. We will show that the statement is also true for $n = k$.

Letting $z = z + \tau_k$ in the equation (3) (in which $n = k$) and dividing through the equation by $e^{\alpha_k \tau_k}$, we have

$$\sum_{i=1}^{k-1} H_i(z + \tau_k)\, e^{\alpha_i (z+\tau_k) - \alpha_k \tau_k} + H_k(z)\, e^{\alpha_k z} = 0 . \tag{4}$$

Eliminating $H_k(z)\, e^{\alpha_k z}$ from equations (4) and (3), we get

$$\sum_{i=1}^{k-1} \{H_i(z) - C_i H_i(z + \tau_k)\}\, e^{\alpha_i z} = 0 \tag{5}$$

where $C_i = e^{(\alpha_i - \alpha_k)\tau_k}$, $i = 1, 2, \cdots, k-1$. From this, by the induction hypothesis, we conclude that

$$H_i(z) - C_i H_i(z + \tau_k) = \sum_{j=1}^{k-2} d(i, j)\, e^{l(i,j)\gamma_i z}, \qquad i = 1, 2, \cdots, k-1 \tag{6}$$

where $l(i, j)$; $i = 1, 2, \cdots, k-1$ and $j = 1, 2, \cdots, k-2$ are integers and $d(i, j)$; $i = 1, 2, \cdots, k-1$, $j = 1, 2, \cdots, k-2$, and γ_i are constants. We note here that $e^{\gamma_i z}$ has a period τ_i.

Now it is easy to verify (see e.g. [2, Chap. XIII, § 1]) that the general entire solution of the linear difference equation has the form:

$$H_i(z) = a_i(z)\, e^{\beta_i z} + \sum_{j=1}^{k-2} b(i, j)\, e^{l(i,j)\gamma_i z} \tag{7}$$

where $a_i(z)$ is a periodic entire function with period τ_k and β_i and $b(i, j)$, j, $j = 1, 2, \cdots, k-2$, $i = 1, 2, \cdots, k-1$ are constants.

From this we have

$$\left\{ H_i(z) - \sum_{j=1}^{k-2} b(i, j)\, e^{l(i,j)\gamma_i z} \right\} = a_i(z)\, e^{\beta_i z}. \tag{8}$$

It follows again from the key lemma that

$$(9) \qquad H_i(z) - \sum_{j=1}^{k-2} b(i,j)\, e^{l_j \gamma_i z} = b(i, k-1)\, e^{\delta_i z}$$

where $b(i, k-1)$ and δ_i are constants.

By the periodicity of $H_i(z)$ we can conclude that $\delta_i = l(i, k-1)\gamma_i$ for some integer $l(i, k-1)$. So we have

$$(10) \qquad H_i(z) = \sum_{j=1}^{k-1} b(i,j)\, e^{l(i,j)\gamma_i z}; \qquad i = 1, 2, \cdots, k-1.$$

By exactly the same argument, if we eliminate $H_i(z)$ instead of $H_k(z)$ from equations (4) and (3), we will get

$$(11) \qquad H_k(z) = \sum_{j=1}^{k-1} b(k,j)\, e^{l(k,j)\gamma_k z}.$$

Thus the statement is valid for $n = k$. Therefore, the theorem is established.

REFERENCES

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